MATRIX

Q. It is the anengment of elements on entries in reectangular arrey or row wise/column wise in a closed bracket parentesis.

Ex
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
→ Row
→ Column

Eg - The above example is not arrenged in rectangular arrey from so it not a example of matrix.

Ex
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{bmatrix}$$
→ This is not matrix

Order of matrix

If the metrix as 'M' number of rows & 'n' no of column the order of matrix is m by n (MXn)

Ex $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ 2X2 $\begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 9 \end{bmatrix}$ 2X3 $\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$ 3X1 $\begin{bmatrix} 1 & 2 & 6 & 2 \end{bmatrix}$ 1X4

Type of matrix

1. Row Matrix ÷ If the element of the matrix is arrenged in raw ~~If the element~~ wise on a singel line them the matrix is called Row matrix.

The order of a row matrix is 1XM where m is the number of column.

Ex. $\begin{bmatrix} 2 & 7 & 6 & 9 & 4 \end{bmatrix}$ 1X5

2. Column matrix? If the element are arrengment in a column wise are is vatical single line in a matrix then the matrix is said to be column matrix.

Ex $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \end{bmatrix}$ 4X1

The order of column matrix is $M \times 1$ where M is the number of Row.

3. Zero Matrix :- The matrix in which the number of row equal the number of column are called square matrix.

Ex $\begin{bmatrix} 3 & 2 & 5 \\ 4 & 2 & 6 \\ 1 & 2 & 4 \end{bmatrix}$ 3×3

4. Diagonal matrix :- The matrix in which Diagonal element of number and other element are zero that is called Diagonal matrix.

Ex $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ 3×3

5. Scalar matrix :- The matrix in which Diagonal element number are same and other element are zero.

Ex $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 3×3

6. Unite Matrix (identify) :-
The matrix in which Diagonal element are 1 and 0 other element are zero them are denote by that is called unite matrix.

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3×3

7. Rectongulan Matrix :-

The matrix in which the number of Rows not equal number of columns that is called Rectongulan matrix.

Ex $\begin{bmatrix} 3 & 2 & 5 \\ 5 & 6 & 5 \end{bmatrix}$ 2×3

8. ~~Transpose of matrix~~ :

~~The matrix in which rows column are interchange equal number of columns that is called Rectangular Matrix.~~

~~Ex $\begin{bmatrix} 3 & 2 & 5 \\ 5 & 6 & 5 \end{bmatrix}_{2\times3}$~~

8. Transpose of matrix :

The matrix in which rows column are interchanged the rows into changed columns and the columns into changed into rows is denote by :

Ex-

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 5 & 6 \\ 1 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 1 \\ 2 & 5 & 6 \\ 5 & 3 & 0 \end{bmatrix}$$

9. **Equal matrix** :- The matrix in which two matrix and one is alphabets and one is number. That is called equal matrix and order same

Ex - $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix}$ $A = 3$, $B = 5$, $C = 1$, $D = 5$

10. Symmetrix matrix :-

The matrix in term as ~~syst~~ symmetrix is equal to transpose in other words of $A = A'$ the matrix A called symmetrix.

Ex-

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \qquad A' = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & e \end{bmatrix}$$

11. Skew Symmetrix matrix:-

A matrix is called skew symmetrix. It is negative is equal transpose. In the words If $A' = A$ that matrix A called skew symmetrix.

Ex-

$$A = \begin{bmatrix} 1 & +5 \\ -5 & 4 \end{bmatrix} \quad A' = \begin{bmatrix} 1 & -5 \\ 5 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & d & -e \\ -d & 0 & -f \\ e & -f & 0 \end{bmatrix} \quad A' = \begin{bmatrix} 0 & -d & e \\ d & 0 & -f \\ -e & f & 0 \end{bmatrix}$$

12. Sub Matrix:- The matrix in which deleting some rows or colums are called sub matrix.

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 6 \\ 7 & 7 & 8 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 \\ 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ 7 & 7 & 8 \end{bmatrix}$$

13. Singular matrix:- The matrix in which whose determinant zero that is singular matrix.

Ex-
$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

14. Non-singular Matrix:- The matrix in which whose determinant not equal that is called Non-singular matrix.

Ex-
$$\begin{bmatrix} 5 & 6 \\ 2 & 8 \end{bmatrix}$$

15. Orthogonal matrix:-

$A' \times A = I$

$A \times A' = I$

Algebra of Matrix:-

* Equality of Matrix:- The matrix A & B are equal matrix If
  (i) The order of both the matrix should be same.

(ii) Each respective element A is same as that of B.

Matrix addition:

Matrix addition is defined if, and only if matrices have same order.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$A+B = \begin{bmatrix} a_{11}+b_{11}, & a_{12}+b_{12}, & a_{13}+b_{13} \\ a_{21}+b_{21}, & a_{22}+b_{22}, & a_{23}+b_{23} \\ a_{31}+b_{31}, & a_{32}+b_{32}, & a_{33}+b_{33} \end{bmatrix}$$

Note: The order of resultant matrix is same as that of parental matrix.

properties:

(i) Matrix addition satisfy commutative law.

$$A+B = B+A$$

Ex $A = \begin{bmatrix} 4 & 5 & -9 \\ 2 & -2 & 4 \\ 3 & 2 & 1 \end{bmatrix}_{3\times3}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 2 \\ -3 & 2 & -1 \end{bmatrix}_{3\times3}$

$$A+B = \begin{bmatrix} 4 & 5 & -9 \\ 2 & -2 & 4 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 2 \\ -3 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & -9 \\ 4 & 2 & 6 \\ 0 & 4 & 0 \end{bmatrix}$$

(ii) Addition matrix hold associative law let A, B, C are matrices of same order then, $(A+B)+C = A+(B+C)$

Ex -

$A = \begin{bmatrix} -1 & 4 & 2 \\ 2 & 4 & 9 \\ -1 & -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 4 & 9 \\ 2 & -2 & 4 \\ -2 & 4 & -6 \end{bmatrix}$

$$(A+B)+C = \left(\begin{bmatrix} -1 & 4 & 2 \\ 2 & 4 & 9 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix} \right) + \begin{bmatrix} -1 & 4 & 9 \\ 2 & -2 & 4 \\ -2 & 4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 & 2 \\ 4 & 4 & 9 \\ 0 & -2 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 9 \\ 2 & -2 & 4 \\ -2 & 4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 12 & 11 \\ 6 & 2 & 13 \\ -2 & 2 & 0 \end{bmatrix}$$

$$A+(B+C) = \begin{bmatrix} -1 & 4 & 2 \\ 2 & 4 & 9 \\ -1 & -1 & 4 \end{bmatrix} -1 \left(\begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 9 \\ 2 & -2 & 4 \\ -2 & 4 & -6 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & 4 & 2 \\ 2 & 4 & 9 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 8 & 9 \\ 4 & -2 & 4 \\ -1 & +3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 12 & 11 \\ 6 & 2 & 13 \\ -2 & 2 & 0 \end{bmatrix}$$

iii. Null matrix or zero matrix is the additive identify of all matrices.

$A+0=A$ (A and 0 are matrices of same order)

Solution-1

$A = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A+0 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow A+0 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

(3) Matrix Subtraction :-

let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{m \times n}$

and subtraction is represent as 'A-B'

$A-B = [a_{ij}] - [b_{ij}]$

$A+(-B) = [a_{ij}] + [-b_{ij}]$

Ex- $\begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & -4 \\ 0 & 2 & 9 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 2 & 2 \\ 0 & 4 & 2 \\ 3 & 2 & 9 \end{bmatrix}$

$$A-B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & -4 \\ 0 & 2 & 9 \end{bmatrix} - B = \begin{bmatrix} -2 & 2 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(-2) & 4-2 & 2-3 \\ 3-0 & -2-4 & -4-2 \\ 0-3 & 2-2 & 9-9 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 3 & -6 & -6 \\ -3 & 0 & 0 \end{bmatrix}$$

4. Matrix multiplication:

Matrix multiplication is possible if and only if no. of column of left metrix is same as that of right matrix.

let $A = [a_{ij}]_{m \times n}$ $B = [b_{ij}]_{m \times n}$

Matrix multiplication is represented as

$$A \times B = [a_{ij}] \times [b_{ij}]$$

$$A \times B = [a_{ij} \times b_{ij}]$$

Ex- ① $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e \\ f \end{bmatrix}$

$$AB = \begin{bmatrix} ea + bf \\ ec + df \end{bmatrix}$$

② $A = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix}_{2 \times 2}$ $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2 \times 1}$

$$\Rightarrow AB = \begin{bmatrix} (2 \times 1) + (1 \times 4) \\ (4 \times 4) + (9 \times 4) \end{bmatrix}$$

$$\Rightarrow AB = \begin{matrix} 2+4 \\ 16+36 \end{matrix}$$

$$\Rightarrow AB = \begin{bmatrix} 6 \\ 52 \end{bmatrix}$$

③ $A = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} (2 \times 1) + (1 \times 4) & (2 \times 2) + (1 \times (-2)) \\ (4 \times 4) + (9 \times 4) & (4 \times 2) + (9 \times -2) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4+(-2) \\ 52 & 8+(-18) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 \\ 52 & -10 \end{bmatrix}$$

Scalar multiplication of matrix:-

If $A = [a_{ij}]_{m\times n}$ be the matrix of order $m\times n$ and α be the scalar value which has to be multiplied with 'A' i.e αA would be as.

$A = [a_{ij}]_{m\times n}$

$\boxed{\alpha A = \alpha a_{ij\, m\times n}}$

Ex-①

$A = \begin{bmatrix} 2 & 4 \\ -3 & 9 \end{bmatrix}$

$2A = \begin{bmatrix} 2 & 4 \\ -3 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 2\times2 & 4\times2 \\ -3\times2 & 9\times2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 8 \\ -6 & 18 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 \\ 3 & 9 \end{bmatrix}$ find $8A - 3B$

$8A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 16 & 8 \\ 32 & 16 \end{bmatrix}$

$3B \Rightarrow \begin{bmatrix} 4 & 5 \\ 3 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & 15 \\ 9 & 27 \end{bmatrix}$

$8A - 3B \Rightarrow \begin{bmatrix} 16 & 8 \\ 32 & 16 \end{bmatrix} - \begin{bmatrix} 12 & 15 \\ 9 & 27 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} (16-12) & (8-15) \\ (32-9) & (16-27) \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 23 & -11 \end{bmatrix}$

properties

i) Matrix multiplication is not always commutative

Case-1

If A and B are matrixs of different order Then either AB is defined or BA is defined

Ex- $\begin{bmatrix} 3 & 2 \\ -6 & 4 \\ 1 & 9 \end{bmatrix}_{3\times2}$ $B = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}_{2\times2}$

$AB = \begin{bmatrix} (3\times1)+(2\times3) & (3\times4)+(2\times5) \\ (-6\times1)+(4\times3) & (-6\times4)+(4\times5) \\ (1\times1)+(9\times3) & (1\times4)+(9+5) \end{bmatrix}$

$= \begin{bmatrix} 3+6 & 12+10 \\ -6+12 & -24+20 \\ 1+27 & 4+45 \end{bmatrix}$

$= \begin{bmatrix} 9 & 22 \\ 6 & -4 \\ 28 & 49 \end{bmatrix}$ (here AB is defined but BA is not defined)

Here, AB is defined but BA is defined because the no. of column is matrix B is not equal to no of row of matrix 'A'.

$$AB \neq BA$$

Case

$\therefore$ If A and B are square matrix of same order both AB and BA are defined

$$[A]_{m\times m}[B]_{m\times m} \neq [B]_{m\times m}[A]$$

Ex-(i)
$$\begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} -2 & -1 \\ 2 & 3 \end{bmatrix}_{2\times 2}$$

$$AB = \begin{bmatrix} (1\times(-2))+(3\times 2) & 1\times(-1)+(3\times 3) \\ (4\times(-2))+(9\times 2) & 4\times(-1)+(9\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} -2+6 & -3+9 \\ -8+18 & -4+27 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 23 \end{bmatrix}$$

~~BA =~~

$$BA = \begin{bmatrix} (-2\times 1)+(-1\times 4) & (-2\times 3)+(-1\times 9) \\ (2\times 1)+(3\times 4) & (2\times 3)+(3\times 9) \end{bmatrix}$$

$$= \begin{bmatrix} -2+(-4) & -6+(-9) \\ 2+12 & 6+27 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -15 \\ 14 & 33 \end{bmatrix}$$

$$AB \neq BA$$

Ex (ii)
$$A = \begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1\times 3)+(3\times 0) & (3\times 3)+(9\times 0) \\ (4\times 9)+(9\times 0) & (4\times 0)+(9\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & 9+0 \\ 0+12 & 0+27 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 9 \\ 12 & 27 \end{bmatrix}$$

$$BA = \begin{bmatrix} (3\times 1)+(0\times 4) & (3\times 3)+(0\times 9) \\ (0\times 1)+(3\times 4) & (0\times 3)+(3\times 9) \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & 9+0 \\ 0+12 & 0+27 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 12 & 27 \end{bmatrix} \quad AB = BA \text{ (proved)}$$

(ii) Multiplication holds associative low :

If A.B & C are matrix, then

$(A \times B)C = A(B \times C)$ (AB, BC, ABC difined)

Ex - ① $A = \begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2\times1}$ $C = \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times2}$

$(AB)C = A(BC)$

$(AB)C = \left(\begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \begin{bmatrix} 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2+4 \\ 3+36 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 39 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6 & 18 \\ 39 & 117 \end{bmatrix}$

$A(BC)$

$A(BC) = \begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \end{bmatrix}\right)$

$= \begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}$

$= \begin{bmatrix} 2+4 & 6+12 \\ 3+36 & 9+108 \end{bmatrix}$

$= \begin{bmatrix} 6 & 18 \\ 39 & 117 \end{bmatrix}$

$(AB)C = A(BC)$ (proved)

$$\boxed{[A]_{m\times n} [B]_{n\times p} (C)_{p\times q} = [ABC]_{n\times c}}$$

⑥ For a squar matrix of oder 'n' the multiplicative identify matrix of order 'n' identify of matrix if

Ex - If $A = \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix}_{2\times2}$

multiplicative identify is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

$AI = \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2+0 & 0+3 \\ 4+0 & 0+9 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix}$

$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix}$

$= \begin{bmatrix} 2+0 & 3+0 \\ 0+4 & 0+9 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 4 & 9 \end{bmatrix}$

$\therefore AI = IA =$ (Commutative low)

DETERMINANT :-

The determinat is also denoted by the symbol Δ the system of equation.

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \\ a_3x + b_3y + c_3z + d_3 = 0 \end{cases}$$

admits a solution if $(a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1) \neq 0$

The above expression can be denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ which is called a determinat of order 3

ie $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_3 + a_2b_3c_1 - a_3b_2c_1$$

Minor and cofactor :-

minor : minor of an element a_{ij} of the determinat of matrix A is the determinat obtained by deleting ith row and j ith column, and it is denoted by M_{ij}.

In a determinat $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$ and so on

cofactor : The cofacter of an element a_{ij} defined as $(-1)^{i+j} M_{ij}$ where M_{ij}, is the minor of a_{ij}, it is denoted by c_{ij}.

i.e. c_{ij} = cofacter of a_{ij} = $(-1)^{i+j} (M_{ij})$

$c_{11} = (-1)^{1+1} M_{11} = M_{11}$

$c_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$c_{13} = (-1)^{1+3} M_{13} = M_{13}$

Let $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + 4\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix}$

$= 2(4+3) - 3(-2-12) + 4(1-8)$

$= 2(7) - 3(-14) + 4(-7)$

$= 14 + 42 - 28$

$= 56 + 28$

$= 28$

Properties of determinat

property 1: The value of the determinat is not altered by changing the rows into columns and the coloumn into rows.

i.e. $|A'| = |A|$ where A' = transpose of Matrix A

Ex — $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

LHS $= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

RHS $= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

Ex —

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ as

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$

property 2: If two adjicent rows or columns of a determinat are interchanged then the sign of the determinant changing its numereral value.

Ex —

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$ (changing 1st and 2nd row)

or $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$ (changing 1st and 2nd coloumn)

As $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

$$\Delta' = \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = a_2b_1 - a_1b_2 = -(a_1b_2 - a_2b_1) = -\Delta$$

Ex---

$$\Delta = \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 2-12 = -10$$

$$\Delta' = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12-2 = 10 = -\Delta$$

Property 3: If two rows or two coloumns of a determinat are identical then the value of the determinat is zero

Ex--- (Two rows are equal)

$$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = a_1b_1 - a_1b_1 = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_2 & c_2 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_2 & c_2 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1(b_2c_2 - b_2c_2) - b_1(a_2c_2 - a_2c_2) + (a_2b_2 - a_2b_2) = 0$$

Property 4: If each elements of any row or any column is multiplied by the same factor then the determinat is multiplied by that factor.

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

As, $\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = mad - mbc = m(ad - bc) = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

As. $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ma_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - mb_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + mc_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$$= m\left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3\end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3\end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3\end{vmatrix}\right\}$$

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3\end{vmatrix}$$

Example :-

$$\begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3\end{vmatrix} = 50\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3\end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3\end{vmatrix} = 50\begin{vmatrix} 5 & 1 \\ 1 & 3\end{vmatrix} - 100\begin{vmatrix} 4 & 1 \\ 2 & 3\end{vmatrix} + 150\begin{vmatrix} 4 & 5 \\ 2 & 1\end{vmatrix}$$

$$= 50(15-1) - 100(12-2) + 150(4-10)$$

$$= 50(14) - 100(10) + 150(-6)$$

$$= 700 - 1000 - 900$$

$$= -1200$$

$$\text{RHS} = 50\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3\end{vmatrix} = 50\left\{1\begin{vmatrix} 5 & 1 \\ 1 & 3\end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 2 & 3\end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 2 & 1\end{vmatrix}\right\}$$

$$= 50(14-20-18) = 50 \times -24 = -1200$$

Note : If any two rows or any two columns in a determinat are proportional, then the value of the determinat is also zero.

Property 5 :- If elements of a row or a column in a deteominat can be expressed as the sum of two or more elements, then the given determinat can be expressed as the sum of two or more determinate of the same order.

Example :

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 \\ a_2 + \alpha_2 & b_2\end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2\end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2\end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} a_1+\alpha_1 & b_1 \\ a_2+\alpha_2 & b_2 \end{vmatrix} = (a_1+\alpha_1)b_2 - (a_2+\alpha_2)b_1$$

$$= a_1b_2 + \alpha_1 b_2 - a_2b_1 - \alpha_2 b_1$$

$$= (a_1b_2 - a_2b_1) + (\alpha_1 b_2 - \alpha_2 b_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix} = \text{R.H.S}$$

Property 6: If ~~to~~ two each element of a row or a column of a determint the equimultiples of corresponding elements of other rows (columns) are added, then value of deter minat remain same.

Example:

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} kc & kd \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k\begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k \times 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{R.H.S.}$$

Adjoint of a Matrix:

If A is a ~~Example~~ square matrix, then the transponse of the matrix of which the elements are cofactor of the corresponding elements on A is called the adjoint by Adj A.

Example –

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{cof } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$ where c_{ij} is the cofactor corresponding to the element a_{ij}.

$$\text{Then Adj } A = (\text{cof } A)^T$$

$$= \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Example 1:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

Here $C_{11} = 1$, $C_{12} = -3$, $C_{21} = -2$, $C_{22} = 1$

$$\text{cof } A = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{Adj. } A = (\text{cof } A)^T = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

Example 2:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

Here $C_{11} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1$

$C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -(2-1) = -1$

$C_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$

$C_{12} = -\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -(2-4) = 2$

$C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

$C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

$C_{13} = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$

$C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -(1-4) = 3$

$C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$

$$\text{Adj } A = (\text{cof } A)^T = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 3 \\ 3 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & -1 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

Inverse of a Matrix:

If A and B are two square Matrices of the same order such that $AB = BA = I$

Then B is called the Multiplicate Inverse of A

B is written as A^{-1} or $B = A^{-1}$

Also, A is called the inverse of B and is written as B^{-1} or $A = B^{-1}$

If A is a non-singular Matrix, then A^{-1} exists & the inverse is given by $A^{-1} = \frac{1}{|A|}(\text{Adj } A)$

Proof:

from theorem-1 we have

$$A.(\text{Adj } A) = |A| I$$

$$\text{or, } A\left(\frac{\text{Adj } A}{|A|}\right) = I$$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

A is a non-singular Matrix. Hence A^{-1} exits

we know that

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

Here $C_{11} = 1$ $C_{12} = -1$

$C_{21} = -1$ $C_{22} = 2$

$$\text{cof } A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\text{Adj } A = (\text{Cof } A)^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Example 2:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5 \neq 0$$

Hence A^{-1} exists

Here $C_{11} = 1, C_{12} = -3$

$C_{21} = -2 \quad C_{22} = 1$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

We know that

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{-1}{5} \end{bmatrix}$$

Example 3:

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ then

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 1(1-6) - 1(2-2) + 1(6-1) = -5 - 0 + 5 = 0$$

Here $|A| = 0$

i.e A is a singular matrix

Hence inverse of A does not exists.

Example 4:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$$= 1(2-1) - 2(4-1) + 3(2-1)$$

$$= 1-6+3 = -2 \neq 0$$

Hence A^{-1} exists.

Now

$c_{11} = 1 \quad c_{12} = -3 \quad c_{13} = 1$

$c_{21} = -1 \quad c_{22} = -1 \quad c_{23} = 1$

$c_{31} = -1 \quad c_{32} = 5 \quad c_{33} = -3$

$$\text{Adj } A = (\text{Cof } A)^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{vmatrix}$$

We know that

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{-5}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$$

Cramer's Rule:

Cramer's rule is used in the solution of simultaneous liner equations.

Consinder the equations in two variables

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

~~ving~~ these two equations by using cross multiplicat method we have.

$$\frac{x}{d_1b_2 - d_2b_1} = \frac{y}{a_1d_2 - a_2d_1} = \frac{1}{a_1b_2 - a_2b_1}$$

i.e $\frac{x}{Dx} = \frac{y}{Dy} = \frac{1}{D}$

where,

$$D = a_1b_2 + a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

$$Dx = d_1b_2 - d_2b_1 = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$$

$Dy = a_1 d_2 - a_2 d_1 \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

therefore $x = \frac{dx}{d}$, $y \frac{dy}{d}$

consider the equation in three variables:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Here

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now multiplying D by x, we have

$$xD \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

$$= \begin{bmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{bmatrix} (C_1 \to C_1 + y C_2 + z C_3)$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = Dx$$

or $xD = Dx$

or $\frac{x}{Dx} = \frac{1}{D}$

similarly we can show that $\frac{y}{Dy}, \frac{1}{D} = \frac{Dy}{D}$, $z = \frac{Dz}{D}$

& $\frac{z}{Dz} = \frac{1}{D}$

where

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

therefore $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$

Example 1

consider, $x + 2y = 5$
$3x + y = 7$

Here $D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$

the system admits a solution

$D_x = \begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} = 5 - 14 = -9$

$D_y = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7 - 15 = -8$

By cramer's rule

$x = \frac{D_x}{D} = \frac{-9}{-5}$, $y = \frac{D_y}{D} = \frac{-8}{-5}$

Example 2

consider $x + y = 3$
$2x + 2y = 7$

~~Here~~ $D = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$

The system does not possen a solution

Example 3:

consider $x - 2y + z = 2$
$6x - 9y + z = 1$
$-9x + 12y + z = 4$

Here $D = \begin{vmatrix} 1 & -2 & 1 \\ 6 & -9 & 1 \\ -9 & 12 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} -9 & 1 \\ 12 & 1 \end{vmatrix} + 2\begin{vmatrix} 6 & 1 \\ -9 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & -9 \\ -9 & 12 \end{vmatrix}$$

$$= -21 + 30 - 9 = 0$$

Similarly $D_x = 0$, $D_y = 0$, $D_z = 0$

So, the system has infinite number of solution.

Example = 4

Consider
$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 1$$

Here
$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= 1(36-2) - 2(18-3) + 3(4-12)$$

$$= 34 - 32 - 24$$

$$= 20 \neq 0$$

the system of equations admits a solution

Similarly we have

$$D_x = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = -20$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = -20$$

By Cramer's Rule

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = -20$$

$x = \frac{D_x}{D} = \frac{-20}{-20} = 1$, $y = \frac{D_y}{D} = \frac{-20}{-20} = 1$, $z = \frac{D_z}{D} = \frac{-20}{-20} = 1$

Solution of simultaneous liner equations by Matrix inverse method :

Let us consider two liner equations with two variables

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

The above system of equations can be written as

$AX = B$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

or $X = A^{-1} B$

$= \frac{1}{|A|} (Adj A) B$,

Example 1 :

Consider the following system of liner equations

$3x - 4y = 1$
$2x + y = 8$

The above system can be written as

$AX = B$

where $A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

or, $X = A^{-1} B = \frac{1}{|A|} (Adj A) B$

Here $|A| = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3(-8) = 11 \neq 0$

So, the system of equations admits solution

Here $c_{11} = 1$ $c_{12} = 2$

$c_{21} = 4$ $c_{22} = 3$

$\text{Cof } A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

so that we have

$x = \frac{1}{|A|}(\text{Adj}) B = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 1+32 \\ -2+24 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 33 \\ 22 \end{bmatrix} = \begin{bmatrix} 33/11 \\ 22/11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Hence $x = 3$ & $y = 2$

Example 2:

Let us consider the system of equations

$x - y + z = 2$

$2x + y - 3z = 5$

$3x - 2y - z = 4$

The system of equations can be written in matrix form as

$Ax = B$ where $A = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & 2 & 1 \end{vmatrix}$ $x \begin{vmatrix} x \\ y \\ z \end{vmatrix}$ $B = \begin{vmatrix} 2 \\ 5 \\ 4 \end{vmatrix}$

or $x = A^{-1}B = \frac{1}{|A|}(\text{Adj} A) B$

$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & -3 \\ 3 & -2 & -1 \end{vmatrix} = 1(-1-6) - 1(-3+9) + 1(-4-3) = -7+7-7 = -7 \neq 0$

So A^{-1} exist and the system admits solution.

We have

$c_{11} = -7$ $c_{12} = -7$ $c_{13} = -7$

$c_{21} = 3$ $c_{22} = -4$ $c_{23} = -1$

$c_{31} = 2$ $c_{32} = 5$ $c_{33} = 3$

So that $\text{cof } A = \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix}$

$\text{adj } A = (\text{cof } A)^T \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{bmatrix}$

So that we have

$x = \frac{1}{|A|}(\text{adj} A) B = (1/-1)\begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 2 \\ -7 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$

$$= (1/-7)\begin{bmatrix} -14 & -15 & +8 \\ -14 & -20 & +20 \\ -14 & -5 & +12 \end{bmatrix} = (1/-7)\begin{bmatrix} -21 \\ -14 \\ -7 \end{bmatrix} = \begin{bmatrix} -21/-7 \\ -14/-7 \\ -7/-7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Hence $x=3$, $y=2$, $z=1$

Pratice Questions :

Prove the following :

a)

(i) $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

Solution :

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$= \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} \quad \because (|A| = |A^t|)$$

$$= (-1) \quad C_1 \leftrightarrow C_2 \quad \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \overset{R_1-R_2}{=} \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} \text{ --- (1)}$$

and $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

$$= (-) \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix}$$

$R_2 (-) R_3$

$$= (-)(-) \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \text{ ———— (2)}$$

From (1) and (2)

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

(Proved)

a) i) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc = \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Solution

$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$= abc\left(\frac{1}{abc}\right)\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$= abc\begin{vmatrix} \frac{1+a}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1+b}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1+c}{c} \end{vmatrix}$

$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$

$C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$

$= abc\begin{vmatrix} 0 & 0 & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ -1 & 1 & 1/b \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix}$

$= abc\left[\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\{1-0\}\right]$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ (Prove)

LHS $= abc + bc + ca + ab$

prove the following :-

iv) $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$= \begin{vmatrix} (a+1)(a+2)-(a+2)(a+3) & -1 & 0 \\ (a+2)(a+2)-(a+2)(a+4) & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$

$= \begin{vmatrix} (a+2)\{a+1-a-3\} & -1 & 0 \\ (a+3)\{a+2-a-4\} & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$

$= \begin{vmatrix} (a+2)(-2) & -1 & 0 \\ (a+3)(-2) & -1 & 0 \\ a+2(a+4) & a+4 & 1 \end{vmatrix}$

$= 1\left[(-2)(a+2)(-1)-(-2)(a+3)(-1)\right]$

$= 2(a+2) - 2(a+3)$

$= 2a+4-2a-6$

$= -2$ (proved)

v) $\begin{vmatrix} a+d & a+d+k & a+d+c \\ c & c+b & c \\ d & d+k & d+c \end{vmatrix}$

Solution: $R_1 \to R_1 - R_3$

$= \begin{vmatrix} a & a & a \\ c & c+b & c \\ d & d+k & d+c \end{vmatrix}$

$C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$

$= \begin{vmatrix} 0 & 0 & a \\ -b & b & c \\ -k & k-c & d+c \end{vmatrix}$

$= a[(-b)(k-c) - (-kb)]$

$\Rightarrow a[-b\cancel{k} + bc + \cancel{k}b]$

$= abc$ (proved)

(vi) $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

Solution: $C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$

$= \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-b & a+b \\ b^2-a^2 & c^2-b^2 & a^2+b^2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-b & a+b \\ (b-a)(b+a) & (c-b)(c+b) & a^2+b^2 \end{vmatrix}$

$= (b-a)(c-b) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & a+b \\ b+a & c+b & a^2+b^2 \end{vmatrix}$

$= (b-a)(c-b)[1\{(c+b)-(b+a)\}]$

$= (b-a)(c-b)[c+\cancel{b}-\cancel{b}-a]$

$= (b-a)(c-b)(c-a)$

$\Rightarrow (a-b)(b-c)(c-a)$

$= (a-b)(b-c)(c-a) = (b-c)(c-a)(a-b)$ (proved

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Soln $R_1 \to R_1 - (R_2+R_3)$

$$\Rightarrow \begin{vmatrix} (b+c)-(b+c) & a-(c+a+c) & a-(b+a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$= 0 - (-2c)\{b(a+b) - bc\} + (-2b)\{bc - c(c+a)\}$

$\Rightarrow 2c\{ab + b^2 - bc\} - 2b\{bc - c^2 - ca\}$

$\Rightarrow 2abc + \cancel{2b^2c} - \cancel{2bc^2} - \cancel{2b^2c} + \cancel{2bc^2} + 2abc$

$\Rightarrow 4abc$ (proved)

(ix) $$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+b^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution

$$= abc \begin{vmatrix} \frac{b^2+c^2}{a} & b & c \\ a & \frac{c^2+a^2}{b} & c \\ a & b & \frac{a^2+b^2}{c} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} b^2+c^2 & b^2 & c^2 \\ a^2 & c^2+a^2 & c^2 \\ a^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$C_1 \to C_1 - (C_2+C_3)$

$$= \begin{vmatrix} b^2+c^2-(b^2+c^2) & b^2 & c^2 \\ a^2-(c^2+a^2+c^2) & c^2+a^2 & c^2 \\ a^2-(b^2+a^2+b^2) & b^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2+a^2 & c^2 \\ -2b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$= -b^2\{(-2c^2)(a^2+b^2) - (2b^2c^2)\}$

$+ c^2\{-2b^2c^2 - (-2b^2)(c^2+a^2)\}$

$= (-b^2)\{-2c^2a^2 - \cancel{2b^2c^2} + \cancel{2b^2c^2}\}$

$+ c^2\{-\cancel{2b^2c^2} + \cancel{2b^2c^2} + 2a^2b^2\}$

$= 2a^2b^2c^2 + 2a^2b^2c^2$

$= 4a^2b^2c^2$

(proved)

Q4. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (b-c)(c-a)(a-b)(bc+ca+a$

Solution

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

$C_1 \to C_1 - C_2$
$C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ -c+a & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ -1 & b+c & c^2 \\ -1 & -a & ab \end{vmatrix}$$

$R_2 \to R_2 - R_3$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ 0 & b+c+a & c^2-ab \\ -1 & -a & ab \end{vmatrix}$$

$= (a-b)(b-c)(c-a)\left[+(-1)\{c^2 - ab - c(b+c+a)\}\right]$

$= (a-b)(b-c)(c-a)\left[(-1)\{\cancel{c^2} - ab - bc - \cancel{c^2} - ca\}\right]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

$= (b-c)(c-a)(a-b)(bc+ca+ab)$ (proved)

<u>Pratice</u> <u>Question</u> :-

<u>Prove</u> <u>that</u> :-

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

<u>Solution</u>:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & \overset{1}{\cancel{a+b+c}} \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 1-1 & 1-1 & 1 \\ 2b-b+c+a & b-c-a-2b & 2b \\ 2c-2c & 2c-c+a+b & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -c-a-b & 2b \\ 0 & c+a+b & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

$= (a+b+c)[(a+b+c)^2 - 0]$

$= (a+b+c)^3$ (R.H.S)

Prove that $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x+2)(x-1)^2$

Solution

$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$= \begin{vmatrix} x-1 & 1-1 & 1 \\ 1-x & x-1 & 1 \\ 1-1 & 1-x & x \end{vmatrix}$

$= \begin{vmatrix} x-1 & 0 & 1 \\ -(x-1) & x-1 & 1 \\ 0 & -(x-1) & x \end{vmatrix}$

$= (x-1)^2 \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & x \end{vmatrix}$

$= (x-1)^2 [1(x+1) + (1-0)]$

$= (x-1)^2 (x+1+1) = (x-1)^2 (x+2)$

(proved)

Show that

$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2$

Solution

$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} 2(x+y+z) & x+z+y & y+x+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$

$$= (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$C_1 \to C_1 - C_2 - C_3$

$$= (x+y+z)\begin{vmatrix} 2-1-1 & 1 & 1 \\ z+x-z-x & z & x \\ x+y-y-z & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 0 & 1 & 1 \\ 0 & z & x \\ x-z & y & z \end{vmatrix}$$

$$= (x+y+z)\left[(x-z)(x-z)\right]$$

$$= (x+y+z)(x-z)^2 \text{ (RHS) (proved)}$$

7) Prove that $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

Sol^n

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= abc\begin{vmatrix} 1-1 & a-b & a^2-b^2 \\ 1-1 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= abc(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= abc(a-b)(b-c)\left[(b+c)-(a+b)\right]$$

$$= abc(a-b)(b-c)(b+c-a-b)$$

$$= abc(a-b)(b-c)(c-a) \text{ (RHS)}$$

(proved)

Trigonometry :

The following three different systems of units are used in the mesurment of trigonometrical angles.

Mesurment of an angle :

There are three systems of mesurment of an angle

(i) Sexagesimal system
(ii) Centesimal system
(iii) Circular system

(i) sexagesimal system :

(i) 1 right angle = 90° (90 degree)
(ii) 1° = 60 sexagesimal minutes ($60'$)
(iii) 1 minutes or $1'$ = 60 sexagesimal second or ($60''$)

(ii) Centesimal system :

(i) 1 right angle = 100 grades (100^g)
(ii) 1^g = 100 centesimal minute
(iii) 1 right angle = $90^\circ = 100^g$

(iii) ~~Circ~~ Circular system :

The unit of mesurment of angles in this system is a radian. A radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of that circle & is denoted by 1^c.

$$\frac{\text{Circumference}}{\text{diameter}} = \pi$$

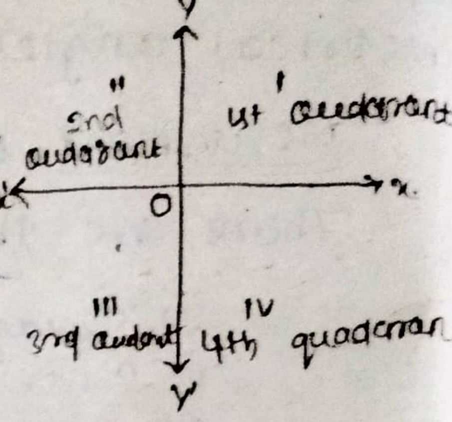
π is a Greek letter, pronounced by "pi". Two right angles = $180^\circ = 200^g = \pi^c$

$$1 \text{ radian} = \frac{2}{\pi} \text{ right angle.}$$

Note : (i) Angle subtended by an arc length is $\theta = \frac{l}{r}$

(ii) The angle subtended at the centre of a circle in radians is 2π radians.

in x - axis , OX and OX' are know as positive & negetive x-axis respectively similililarly . In y - axis , OY & OY' are known as positive & negetive y axis respectively



(fig 2.1)

Now , both axes divide the xy plane in to four equal parts called "quadarants"

(i) XOY is called Ist quadarant .

(ii) $X'OY$ is called 2nd quadarant.

(iii) $X OY'$ is called 3rd quadarant .

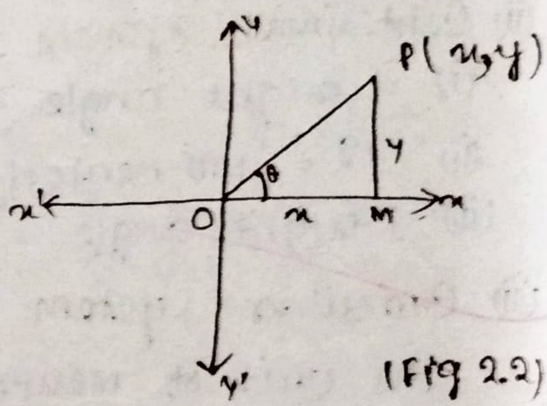
(iv) XOY' is called 4th quadarant.

Let us take a point $P(x,y)$ Draw $PM \perp OX$ in fig 2.2 Join OP

So that $OM = x$, $PM = y$

OPM is a right angle traingle

if θ is an angle mesure such that $0 < \theta < \pi/2$.



(Fig 2.2)

Let $\angle POM = \theta$

So, the side OP opposite of the right angle $\angle PMO$ is known as hypotenuse (h)

The side OM related to right angle and given angle (θ) is base (b) & the side PM is known as the perpendicular (P) .

Now in the right angle traingle ΔOPM .

(1) The ratio of the perpendicular to the hypotenuse is called "sine of the angle θ" and it is written as sin θ. i.e $\sin\theta = \frac{\text{Perpendicular}}{\text{hypotenuse}} = \frac{PM}{OP}\ \left[\frac{P}{h}\right]$

(2) The ratio of the base to the hypotenuse is called "cosine of the angle θ" & it is written as cos θ

i.e $\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{PM}{OP}\ \left[\frac{B}{h}\right]$

(3) The ratio of the perpendicular to the base is called "tangeat of the angle θ" & it is written as tan θ. i.e $\tan\theta = \frac{\text{perpendicular}}{\text{Base}} = \frac{PM}{OM}\ \left[\frac{P}{b}\right]$

(4) The ratio of the hypotenuse to perpendicular is called "cosec cant of the angle θ" & it is written as cosec θ.

i.e $\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{perpendiculers}} = \frac{OP}{PM} = \left[\frac{h}{P}\right]$

(5) The ~~ratie~~ of the hypotenuse to base is called "Seccant of the angle θ" & it is written as sec θ

i.e $\sec\theta = \frac{\text{hypotenuse}}{\text{Base}} = \frac{OP}{OM} = \left[\frac{h}{b}\right]$

(6) The ratio of the base to perpendicular is called "cotangent of the angle θ" & it is written as cot θ

i.e $\cot\theta = \frac{\text{Base}}{\text{perpendicular}} = \frac{OM}{PM}\ \left[\frac{B}{P}\right]$

Notes :

(i) All the above six ratios are called trigonometrical ratio

(ii) $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ or $\sin\theta$ and $\operatorname{cosec}\theta$ are reciprocal ratios

$\sec\theta = \frac{1}{\cos\theta}$ or $\cos\theta$ & $\sec\theta$ are reciprocal ratios.

$\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta$ & $\cot\theta$ are reciprocal ratios.

Trigonometric Functions :

The six trigonometric functions are given by the following.

(i) sine : $R \to [-1, 1]$

(II) cosine : $R \to [-1, 1]$

(III) tangent : $R - \{(2n+1)\frac{\pi}{2} : n \in Z\} \to R$

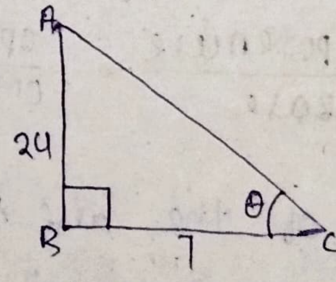
(IV) cotangent : $R - (n\pi : n \in Z) \to R$

(V) secant : $R - \{(2n+1)\frac{\pi}{2} : n \in Z\} \to R - (-1, 1)$

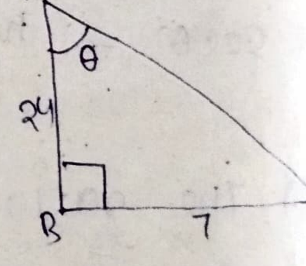
(VI) cosecant : $R - \{n\pi - n \in Z\} \to R - (-1, 1)$

Example - 1

In ΔABC, right angle is at B & AB = 24 cm, BC = 7



(Fig 2.5)



(fig 2.6)

(i) In fig - 1 $\angle B = 90°$ & $\angle C = \theta$ is the given angle

So, Base = b = BC = 7 cm, Hypotenuse = h = AC

& perpendicular = p = AB = 24 cm.

By using pythagoras theorem $p^2 + b^2 = h^2$

$\therefore h^2 = p^2 + b^2 = 24^2 + 7^2 = 625$ & $h = 25$ cm

therefore $\sin A = \frac{24}{25}$, $\cos A = \frac{b}{h} = \frac{7}{25}$ & $\tan = \frac{p}{b} = \frac{24}{7}$

(ii) But in Fig-2 $\angle B = 90°$ & $\angle A = \theta$, is the given angle
So, Base $= b = AB = 7cm$, Hypotenuse $= h = AC$
& perpendicular $= p = BC = 7cm$

By using phythagoras theorem, $p^2 + b^2 = h^2$

$\therefore h^2 = 24^2 + 7^2 = 625$ & $h = 25cm$

(iii) Therefore $\sin A = \frac{p}{h} = \frac{7}{25}$, $\cos A = \frac{b}{h} = \frac{24}{25}$ & $\tan A = \frac{p}{b} = \frac{7}{24}$

Example-2: Let $\cot\theta = \frac{7}{8}$,

So, $\cot\theta = \frac{b}{p} = \frac{7}{8} = k$, where k is a proportionally constant

$\therefore b = 7k$ & $p = 8k$

By using pythagoras theoram: $p^2 + b^2 = h^2$

$\Rightarrow (8k)^2 + (7k)^2 = h^2$

$\Rightarrow h^2 = 113k^2$ or $h = \sqrt{113}k$

Hence, $\sec\theta = \frac{h}{b} = \frac{\sqrt{113}k}{7k} = \frac{\sqrt{113}}{7}$ & $\operatorname{cosec}\theta = \frac{h}{p} = \frac{\sqrt{113}}{8}$

Trigonometry Identity:

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta - \tan^2\theta = 1$

(iii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

proof:

(i) LHS: $\sin^2\theta + \cos^2\theta = (\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{p}{h}\right)^2 + \left(\frac{b}{h}\right)^2$

$= \frac{p^2}{h^2} + \frac{b^2}{h^2} + \frac{p^2 + b^2}{h^2} = \frac{h^2}{h^2} = RHS$

[Note: By pythagoras theorem, In a right angled triangle $p^2 + b^2 = h^2$]

(ii) LHS $= \sec^2\theta - \tan^2\theta = (\sec\theta)^2 - (\tan\theta)^2 = \left(\frac{h}{b}\right)^2 - \left(\frac{p}{b}\right)^2$

$= \frac{h^2}{b^2} - \frac{p^2}{b^2} = \frac{h^2 - p^2}{b^2} = \frac{b^2}{b^2} = 1$ (RHS)

(iii) LHS $= \operatorname{cosec}^2\theta - \cot^2\theta = (\operatorname{cosec}\theta)^2 - (\cot\theta)^2 = \left(\frac{h}{p}\right)^2 - \left(\frac{b}{p}\right)^2$

$\left\{\frac{h^2}{p^2}\right\} - \frac{b^2}{p^2} = \frac{h^2 - b^2}{p^2} = \frac{p^2}{p^2} = 1$ (RHS)

Some Solved Problems :

Q:1 : prove : $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

LHS : $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$

$= \dfrac{\sin\theta + \sin\theta + (1+\cos\theta)(1+\cos\theta)}{(1+\cos\theta)\sin\theta}$

$= \dfrac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)\sin\theta}$

$= \dfrac{\sin^2\theta + 1 + \cos^2\theta + 2\cos\theta}{(1+\cos\theta)\sin\theta}$ $[\because \sin^2\theta + \cos^2\theta = 1]$

$= \dfrac{1+1+2\cos\theta}{(1+\cos\theta)\sin\theta}$

$= \dfrac{2}{\sin\theta} = 2\operatorname{cosec}\theta = \text{RHS}$

Q-2 : prove $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

proof :

LHS $= \dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x}$

$= \dfrac{1(1+\sin x) + 1(1-\sin x)}{(1-\sin x)(1+\sin x)}$

$= \dfrac{1+\sin x + 1 - \sin x}{1-\sin^2 x}$ $[\because (a-b)(a+b) = a^2 - b^2]$

$= \dfrac{2}{\cos^2 x}$ $[\because 1-\sin^2 x = \cos^2 x]$

$= 2\sec^2 x = \text{RHS}$

Q-3: prove $(\operatorname{cosec}\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

proof:

$$LHS := (\operatorname{cosec}\theta - \cot\theta)^2$$

$$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} \quad [\because \sin^2\theta = 1-\cos^2\theta = (1-\cos\theta)(1+\cos\theta)]$$

$$= \frac{1-\cos\theta}{1+\cos\theta} = RHS \text{ (proved)}$$

Q-4: prove $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - 1} + \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta + 1} = 2\sec^2\theta$

proof:

$$LHS = \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - 1} + \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta + 1}$$

$$= \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta + 1) + \operatorname{cosec}\theta(\operatorname{cosec}\theta - 1)}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$$

$$= \frac{\operatorname{cosec}^2\theta + \cancel{\operatorname{cosec}\theta} + \operatorname{cosec}^2\theta - \cancel{\operatorname{cosec}\theta}}{\operatorname{cosec}^2\theta - 1}$$

$$= \frac{2\operatorname{cosec}^2\theta}{\cot^2\theta} = \frac{2}{\sin^2\theta} \cdot \frac{\sin^2\theta}{\cos^2\theta} = 2\sec^2\theta = RHS.$$

Question-5: prove $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

Proof: LHS $= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta}} = \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$

$$= \sqrt{\frac{(1+\sin^2\theta)^2}{\cos^2\theta}}$$

$$= \sqrt{\left(\frac{1+\sin\theta}{\cos\theta}\right)^2} = \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$$

RHS

(proved)

Question - 6

prove: $\frac{\tan x}{1-\cot x} + \frac{\cot x}{1-\tan x} = \sec x \cdot \operatorname{cosec} x + 1$

proof:

LHS: $\frac{\tan x}{1-\cot x} + \frac{\cot x}{1-\tan x}$

$= \frac{\sin x}{\cos x}\frac{1}{1-\frac{\cos x}{\sin x}} + \frac{\cos x}{\sin x}\frac{1}{1-\frac{\sin x}{\cos x}}$

$= \frac{\sin x}{\cos x}\frac{1}{\frac{\sin x-\cos x}{\sin x}} + \frac{\cos x}{\sin x} \cdot \frac{1}{\frac{\cos x-\sin x}{\cos x}}$

$= \frac{\sin^2 x}{\cos x(\sin x-\cos x)} - \frac{\cos^2 x}{\sin x(\sin x-\cos x)}$

$= \frac{\sin^2 x-\cos^2 x}{\sin x\cos x(\sin x-\cos x)}$

$[\because \sin^3 x-\cos^3 x,\ a^3-b^3=(a-b)(a^2+ab+b^2)$

$=(\sin x-\cos x)(\sin^2 x+\cos^2 x+\sin x.\cos x)$

$=(\sin x-\cos x)(1+\sin x.\cos x)$

$=\frac{(\sin x-\cos x)(1-\sin x.\cos x)}{\sin x\cos x(\sin x-\cos x)}$

$= \frac{1}{\sin x\cos x} \cancel{+}+1 = \sec x \operatorname{cosec} x+1$ (RHS) proved

Q.7 prove: $\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta} = 1-2\sec\theta\cdot\tan\theta+2\tan^2\theta$

proof = LHS: $\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta}$

$= \frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta}\cdot\frac{\sec\theta-\tan\theta}{\sec\theta-\tan\theta}$

$= \frac{(\sec\theta-\tan\theta)^2}{\sec^2\theta-\tan^2\theta}$ $[\because \sec^2\theta-\tan^2\theta=1]$

$= \frac{\sec\theta-\tan\theta}{1}$

$= (1+\tan^2\theta)+\tan^2\theta - 2\sec\theta.\tan\theta$

$= 1-2\sec\theta\cdot\tan\theta+2\tan^2\theta$ = RHS

Question - 8 prove that $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta \cdot \cos^2\theta)$

proof

LHS : $\sin^8\theta - \cos^8\theta$

$= (\sin^4\theta)^2 - (\cos^4\theta)^2$

$= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$ [As $a^2 - b^2 = (a-b)(a+b)$]

$= \{(\sin^2\theta)^2 - (\cos^2\theta)^2\}\{(\sin^2\theta)^2 + (\cos^2\theta)^2\}$

$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\}$

[$\because a^2 + b^2 = (a+b)^2 - 2ab$ & $\sin^2\theta + \cos^2\theta = 1$]

$= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) =$ RHS (proved)

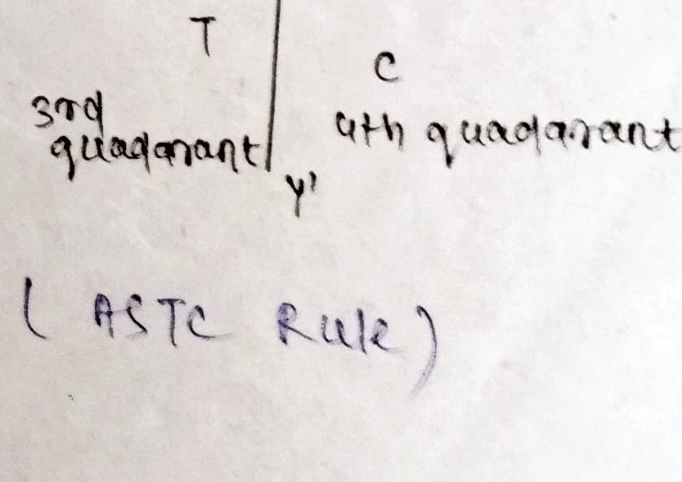
Note : ASTC Rule

(i) In 1st quadrant all T-ratios are +ve.

(ii) In 2nd quadrant sine is +ve & all others -ve.

(iii) In 3rd quadrant tangent is +ve & all others -ve.

(iv) In 4th quadrant cosine is +ve & all others -ve



(ASTC Rule)

	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$\frac{1}{-\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
Sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	∞

Some solved problems

Q1 = find the value of $\frac{5\sec^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ}$

Sol^n:

Using the trigonometry values.

$$\frac{5\sec^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ} = \frac{5\left(\frac{1}{2}\right)^2 + 4\left(\frac{2}{\sqrt{3}}\right)^2 - (1)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\frac{5}{4} + \frac{16}{3} - 1}{\frac{1}{4} + \frac{3}{4}} = \frac{\frac{15+64-12}{12}}{\frac{4}{4}} = \frac{67}{12}$$

Question -2: find the value of $\frac{\cot 45^\circ}{\sqrt{1-\cot^2 60^\circ}}$

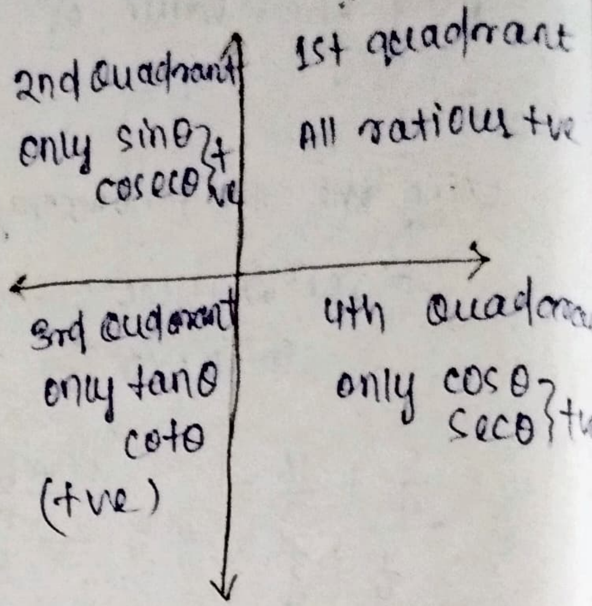
Sol^n

As we know that $\cot 45 = 1$ & $\cot 60^\circ = \frac{1}{\sqrt{3}}$

therefore, $\frac{\cot 45^\circ}{\sqrt{1-\cot^2 60^\circ}} = \frac{1}{\sqrt{1-(1/\sqrt{3})^2}} = \frac{1}{\sqrt{2/3}} = \frac{\sqrt{3}}{\sqrt{2}}$

: Mathematics formula :

1. $\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$
2. $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}}$
3. $\tan\theta = \dfrac{\text{perpendicular}}{\text{Base}}$
4. $\cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}}$
5. $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Base}}$
6. $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{perpendicular}}$



Relation Between Trigonometric Ratios :-

1. $\sin^2\theta + \cos^2\theta = 1$
   $\sin^2\theta = 1 - \cos^2\theta$
   $\cos^2\theta = 1 - \sin^2\theta$
2. $\text{cosec}^2\theta - \cot^2\theta = 1$
   $\text{cosec}^2\theta = 1 + \cot^2\theta$
   $\cot^2\theta = \text{cosec}^2\theta - 1$
3. $\sec^2\theta - \tan^2\theta = 1$
   $\sec^2\theta = 1 + \tan^2\theta$
   $\tan^2\theta = \sec^2\theta - 1$

Angle	0	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Some Important trigonometry Formula :-

(1) $\sin(A+B) = \sin A.\cos B + \cos A.\sin B$

(2) $\sin(A-B) = \sin A.\cos B - \cos A.\sin B$

(3) $\cos(A+B) = \cos A.\cos B - \sin A.\sin B$

(4) $\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

(5) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$

(6) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A.\tan B}$

(7) $\cot(A+B) = \dfrac{\cot A.\cot B - 1}{\cot B + \cot A}$

(8) $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$

(9) $\sin(A+B+C) = \sin A \cdot \cos B \cdot \cos C + \sin B \cdot \cos C \cdot \cos A + \sin C \cdot \cos A$
$\cos B - \sin A \cdot \sin B \cdot \sin C$

(10) $\cos(A+B+C) = \cos A \cdot \cos B \cdot \cos C - \sin A \cdot \sin B \cdot \cos C - \sin A \cdot \cos B \cdot$
$\sin C - \cos A \cdot \sin B \cdot \sin C$

(11) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C}{1 - \tan A \cdot \tan B - \tan B \tan C - \tan \cdot \tan A}$

(12) $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$

(13) $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$

(14) $\sin(A+B) + \sin(A-B) = 2\sin A \cdot \cos B$

(15) $\sin(A+B) - \sin(A-B) = 2\cos A \cdot \sin B$ | $\sin(A-B) - \sin(A+B) = -2\cos A \sin B$

(16) $\cos(A+B) + \cos(A-B) = 2\cos A \cdot \cos B$

(17) $\cos(A-B) - \cos(A+B) = 2\sin A \cdot \sin B$

(18) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(19) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(20) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(21) $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

(22) $\sin 2A = 2\sin A \cdot \cos A$ or $\left(\sin A = 2\sin\frac{A}{2}, \cos\frac{A}{2}\right)$

(23) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ or $\left(\sin A = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}\right)$

(24) $\cos 2A = \cos^2 A - \sin^2 A$ or $\left(\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}\right)$

(25) $\cos 2A = 1 - 2\sin^2 A$ or $2\sin^2 A = 1 - \cos 2A$ or $\sin^2 A = \dfrac{1-\cos 2A}{2}$
or $\left(1 - \cos A = 2\sin^2\frac{A}{2}\right)$ or $\left(\sin^2\frac{A}{2} = \dfrac{1-\cos A}{2}\right)$

(26) $\cos 2A = 2\cos^2 A - 1$ or $2\cos^2 A = 1 + \cos 2A$ or $\cos^2 A =$
$= \dfrac{1+\cos 2A}{2}$ or $\left(1 + \cos A = 2\cos^2\frac{A}{2}\right)$ or $\left(\cos^2\frac{A}{2} = \dfrac{1+\cos A}{2}\right)$

(27) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ or $\cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(28) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ or $\tan A = \dfrac{2\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(29) $\sin 3A = 3\sin A - 4\sin^3 A$

(30) $\cos 3A = 4\cos^3 - 3\cos A$

(31) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

1) Now

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

similarly

$\cot(-\theta) = -\cot\theta$

$\sec(-\theta) = \sec\theta$

$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

Example :

$\sin(-60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

$\cos(-30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\tan(-45^\circ) = -\tan 45^\circ = -1$

(2) T- ratios of $(90-\theta)$ in terms of θ, for

$\sin(90^\circ-\theta) = \cos\theta$

$\cos(90^\circ-\theta) = \sin\theta$

$\tan(90^\circ-\theta) = \cot\theta$

similarly,

$\cot(90-\theta) = +\tan\theta$

$\sec(90-\theta) = +\operatorname{cosec}\theta$

$\operatorname{cosec}(90-\theta) = +\sec\theta$

Example :

$\sin(90^\circ-30^\circ) = +\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\cos(90^\circ-60^\circ) = +\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\tan(90^\circ-45^\circ) = +\cot 45^\circ = 1$

3. T-ratios of $(90^\circ+\theta)$ in terms of θ for

Now,

$\sin(90^\circ+\theta) = \cos\theta$

$\cos(90^\circ+\theta) = -\sin\theta$

$\tan(90^\circ+\theta) = -\cot\theta$

Similarly

$\cot(90° + \theta) = -\tan\theta$

$\sec(90° + \theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(90° + \theta) = +\sec\theta$

Examples :

$\sin 120° = \sin(90° + 30°) = +\cos 30° = \frac{\sqrt{3}}{2}$

$\cos 150° = (\cos(90° + 30°) = + \sin 60° = -\frac{\sqrt{3}}{2}$

$\tan 135 = (\tan(90° + 45°) = -\cot 45° = -1$

$\sec 150° = \sec(90° + 60°) = -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}$

4. T-ratios of $(180° - \theta)$ in terms of θ for all the value.

$\sin(180° - \theta) = +\sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

$\cot(180° - \theta) = -\cot\theta$

$\sec(180° - \theta) = -\sec\theta$

$\operatorname{cosec}(180° - \theta) = +\operatorname{cosec}\theta$

Example :

$\cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\sec 120° = \sec(180° - 60°) = -\sec 60° = -2$

$\cot 135° = \cot(180° - 45°) = -\cot 45° = -1$

5. T-ratios of $(180° + \theta)$ in term of θ.

$\sin(180° + \theta) = -\sin\theta$

$\cos(180° + \theta) = -\cos\theta$

$\tan(180° + \theta) = +\tan\theta$

$\cot\theta(180° + \theta) = +\cot\theta$

$\sec\theta(180° + \theta) = -\sec\theta$

$\operatorname{cosec}\theta(180° + \theta) = -\operatorname{cosec}\theta$

Examples :

$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$

$\cos 225° = \cos(180° + 45°) = -\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 240° = \tan(180° + 60°) = +\tan 60° = \sqrt{3}$

6. T- Ratios of $(270-\theta)$ in terms of θ.

$\sin(270^\circ-\theta) = -\cos\theta$
$\cos(270^\circ-\theta) = -\sin\theta$
$\tan(270^\circ-\theta) = +\cot\theta$
$\cot(270^\circ-\theta) = +\tan\theta$
$\sec(270^\circ-\theta) = -\operatorname{cosec}\theta$
$\operatorname{cosec}(270-\theta) = -\sec\theta$

Examples :

$\sin 210^\circ = \sin(270-60^\circ) = -\cos 60^\circ = -\frac{1}{2}$
$\cos 240^\circ = \cos(270^\circ-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$
$\tan 225^\circ = \tan(270-45^\circ) = +\cot 45^\circ = +1$

7. T- ratios of $(270^\circ+\theta)$ in terms of θ.

$\sin(270^\circ+\theta) = -\cos\theta$
$\cos(270^\circ+\theta) = +\sin\theta$
$\tan(270^\circ+\theta) = -\cot\theta$
$\cot(270^\circ+\theta) = -\tan\theta$
$\sec(270^\circ+\theta) = +\operatorname{cosec}\theta$
$\operatorname{cosec}(270^\circ+\theta) = -\sec\theta$

Examples :

$\sin 315^\circ = \sin(270^\circ+45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$
$\cos 300^\circ = \cos(270^\circ+30^\circ) = \sin 30^\circ = \frac{1}{2}$
$\cot 330^\circ = \cot(270+60^\circ) = -\tan 60^\circ = -\sqrt{3}$

8. T- ratios of $(360^\circ-\theta)$ in terms of θ:

$\sin(360^\circ-\theta) = -\sin\theta$
$\cos(360^\circ-\theta) = +\cos\theta$
$\tan(360^\circ-\theta) = -\tan\theta$
$\cot\theta(360^\circ-\theta) = -\cot\theta$
$\sec(360^\circ-\theta) = +\sec\theta$
$\operatorname{cosec}(360^\circ-\theta) = -\operatorname{cosec}\theta$

Example :

$\sin 330^\circ = \sin(360^\circ-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$
$\sec 300 = \sec(360^\circ-60^\circ) = \sec 60^\circ = 2$
$\operatorname{cosec} 315^\circ = \operatorname{cosec}(360^\circ-45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}$

9. T-ratios of $(360°+\theta)$ in terms of θ. For all the values:

$\sin(360°+\theta) = +\sin\theta$
$\cos(360°+\theta) = +\cos\theta$
$\tan(360°+\theta) = +\tan\theta$
$\cot(360°+\theta) = +\cot\theta$
$\sec(360°+\theta) = +\sec\theta$
$\operatorname{cosec}(360°+\theta) = +\operatorname{cosec}\theta$

Some Solved problems :

Q.1 = State $\cos 302°$ is (+ve) or (-ve).

Sol^n

$\cos(3\times 90° + 32°) = \sin 32°$

$\because$ $\sin 32°$ lies in 1st quadarant & by ASTC rule all the t-ratios are positive in 1st quadarant so, $\cos 32°$ is positive sign.

Q.2 : Find the value of $\sin 1230°$.

Sol^n

$\sin 1230 = (\sin \overset{3\times}{}360° + 150°) = \sin 150°$

$= \sin(180° - 30°) = \sin 30° = 1/2$

Q.3 : Express $\sin 1185°$ as the trigonometric ratio of some acute angle.

Sol^n :

$\sin 1185° = (\sin 13\times 90° + 15°)$

$= (-1)^{\frac{13-1}{2}} \cos 15° = (-1)^{6} \cos 15° = \cos 15°$

Q.4 : Find the value of $\log \tan 17° + \log \tan 37° + \log \tan 53° + \log \tan 73°$.

Sol^n

$\log\tan 17° + \log\tan 37° + \log\tan 53° + \log\tan 73°$

$= \log\tan 17° \tan 37° \tan 53° \tan 73°$

$= \log\tan 17° \tan 37° \tan(90° - 37°) \tan(90° - 17°)$

$= \log \tan 17° \tan 37° \cot 37° \cot 17°$

$= \log 1 = 0$ [$\because \tan 17° \cot 17° = 1$ and $\tan 37° \cot 37° = 1$]

Q:5: Show that $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90-\theta)} = -1$

Proof

$$LHS = \frac{\cos(90+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90-\theta} = \frac{-\sin\theta \cdot \sec\theta(-\tan\theta)}{\sec\theta(-\sin\theta)\tan\theta} = -$$

Theorem-1.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$

(IV) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$.

Theorem-2 :-

(i) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(ii) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

Theorem-3:-

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}$$ for $A, B, C \in R$.

Theorem-4:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

Proof:

(i) $\sin(A+B)\sin(A-B)$

$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - \cancel{\cos^2}(1-\sin^2 A)\sin^2 B$

$= \sin^2A - \sin^2A\,\sin^2B - \sin^2B + \sin^2A\,\sin^2B$

$= \sin^2A - \sin^2B$ (proved).

$= (1-\cos^2A) - (1-\cos^2B)$

$= \cos^2B - \cos^2A$ (proved)

(ii) $\cos(A+B)\cos(A-B) = \cos^2A - \sin^2B = \cos^2B - \sin^2A$

$\cos(A+B)\cos(A-B)$

$= (\cos A.\cos B - \sin A.\sin B)(\cos A\cos B + \sin A.\sin B)$

$= \cos^2A\,\cos^2B - \sin^2A\,\sin^2B$

$= \cos^2A(1-\sin^2B) - (1-\cos^2A)\sin^2B$

$= \cos^2A - \cos^2A\,\sin^2B - \sin^2B + \cos^2A\,\sin^2B$

$= \cos^2A - \sin^2B$ (proved)

$= (1-\sin^2A) - (1-\cos^2B)$

$= \cos^2B - \sin^2A$ (proved)

Some solved Problems:

Q:1: Find the value of $\cos 15°$.

$\cos 15° = \cancel{\cos}(45°-30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Q2) Find the value of $\cos 50°\cos 40° - \sin 50°\sin 40°$

Sol: $\cos 50°\cos 40° - \sin 50°\sin 40° = \cos(50°+40°) = \cos 90° = 0$

Q3) If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$ find the value of $\tan(A+B)$

Sol:

we know that, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{5}{6}}{\frac{5}{6}}$

$= 1$ (Answer)

Q) (4) : prove that $\frac{\cos 9' + \sin 9'}{\cos 9' - \sin 9'} = \tan 54'$

proof

$$LHS = \frac{\cos 9' + \sin 9'}{\cos 9' - \sin 9'} = \frac{\frac{\cos 9'}{\cos 9'} + \frac{\sin 9'}{\cos 9'}}{\frac{\cos 9'}{\cos 9'} - \frac{\sin 9'}{\cos 9'}}$$ (Dividing throughout by $\cos 9'$)

$$= \frac{1 + \tan 9'}{1 - \tan 9'} = \frac{\tan 45' + \tan 9'}{1 - \tan 45' \tan 9'} \quad [\because \tan 45' = 1)$$

$$= \tan(45' + 9') = \tan 54' = RHS$$

Q) - 5 : prove that $\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B) = 0$

Proof : $\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B)$

$= \sin A(\sin B \cos C - \cos B \sin C) + \sin B(\sin C \cos A - \cos C \sin A) + \sin C(\sin A \cos B - \cos A \sin B)$

$= \sin A \sin B \cos C - \sin A \cos B \sin C + \sin B \sin C \cos A - \sin B \cos C \sin A + \sin C \sin - \sin C \cos A \sin B$

(All the terms are cancelled with each other)

$= 0 = R.H.S$

Transformation of a product into a sum or Difference, and vice versa.

(i) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

(ii) $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$

(iii) $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$

(iv) $\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$

proof

From above established theorems 1 & 2

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

Adding (i) and (ii)

$\sin(A+B) + \sin(A-B) = (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B)$

$= 2\sin A \cos B.$

Subtracting (ii) from (i)

$\sin(A+B) - (\sin A-B) = (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)$

$= 2\cos A \sin B$

Again adding (iii) and (iv)

$\cos(A+B) + (\cos(A-B) = (\cos A \cos B + \sin A \sin B) + (\cos A \cos B - \sin A \sin B)$

$= 2\cos A \cos B$

Subtracting (iv) from (iii)

$\cos\cancel{(A+B)} - \cos(A-B) = (\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B)$

$= 2\sin A \sin B$ (proved)

Note

Let $A+B = C$ and $A-B = D$

then $2A = C+D$ or $A = \frac{C+D}{2}$

and $2B = C-D$ or $B = \frac{C-D}{2}$

Putting the above value of A and B in above four formula, we get

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Some solved problems :

Q.1. prove that $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

proof

$LHS = \sin 50^\circ - \sin 70^\circ + \sin 10^\circ$

$= (\sin 60^\circ - 10) - \sin (60^\circ + 10)^\circ + \sin 10^\circ$

$= 2\cos 60^\circ \sin 10^\circ + \sin 10^\circ$ ($\because \sin(A-B) - \sin(A+B) = 2\cos A \sin B$)

$= 2 \times \frac{1}{2} \sin 10^\circ + \sin 10^\circ$

$= -\sin 10^\circ + \sin 10^\circ = 0 = RHS$

Q) 2) prove $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$

$= (\sin 10^\circ + \sin 50^\circ) + (\sin 20^\circ + \sin 40^\circ)$

$= 2\sin\left(\frac{10+50^\circ}{2}\right)\cos\left(\frac{10^\circ - 50^\circ}{2}\right) + 2\sin\left(\frac{20^\circ + 40^\circ}{2}\right)\cos\left(\frac{20^\circ - 40^\circ}{2}\right)$

$= 2\sin 30^\circ \cos 20^\circ + 2\sin 30^\circ \cos 10^\circ$

$= 2 \frac{1}{2} \sin 30^\circ (\cos 20^\circ + \cos 10^\circ)$

$= 2\frac{1}{2}(\cos 20^\circ + \cos 10^\circ)$

$= \cos 20^\circ + \cos 10^\circ$

$= \cos(90^\circ - 70^\circ) \cos(90^\circ - 80^\circ)$

$= \cos 70^\circ + \cos 80^\circ = RHS$

Q) 3) prove that $= \frac{\cos 7a + \cos 3a - \cos 5a - \cos a}{\sin 7\alpha - \sin 3\alpha - \sin 5\alpha + \sin\alpha} = \cot 2\alpha$

proof

$LHS = \frac{\cos 7\alpha + \cos 3\alpha - \cos 5\alpha - \cos\alpha}{\sin 7\alpha - \sin 3\alpha - \sin 5\alpha + \sin\alpha}$

$= \frac{(\cos 7\alpha + \cos 3\alpha) - (\cos 5\alpha + \cos\alpha)}{(\sin 7\alpha - \sin 3\alpha) - (\sin 5\alpha - \sin\alpha)}$

$= \frac{2\cos\left(\frac{7\alpha + 3\alpha}{2}\right)\cos\left(\frac{7\alpha - 3\alpha}{2}\right) - 2\cos\left(\frac{5\alpha + \alpha}{2}\right)\cos\left(\frac{5\alpha - \alpha}{2}\right)}{2\cos\left(\frac{7\alpha + 3\alpha}{2}\right)\cos\left(\frac{7\alpha - 3\alpha}{2}\right) - 2\cos\left(\frac{5\alpha + \alpha}{2}\right)\sin\left(\frac{5\alpha - \alpha}{2}\right)}$

$$= \frac{2\cos 4\alpha \cos 2\alpha - 2\cos 3\alpha \cos 2\alpha}{2\cos 4\alpha \sin 2\alpha - 2\cos 3\alpha \sin 2\alpha}$$

$$= \frac{2\cos 2\alpha (\cos 4\alpha - \cos 3\alpha)}{2\sin 2\alpha (\cos 4\alpha - \cos 3\alpha)}$$

$$= \frac{2\cos 2\alpha}{2\sin 2\alpha} = \cot 2\alpha = RHS$$

Q)4) If $\sin A = k\sin B$, prove that $\tan\frac{1}{2}(A-B) = \frac{k-1}{k+1}\tan\frac{1}{2}(A+B)$

Proof

Given $\sin A = k\sin B$

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{k}{1}$$

$$\Rightarrow \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{k-1}{k+1}$$

$$\Rightarrow \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{k-1}{k+1}$$

$$\Rightarrow \cot\frac{A+B}{2}\tan\frac{A-B}{2} = \frac{k-1}{k+1}$$

$$\Rightarrow \tan\frac{1}{2}(A-B) = \frac{k-1}{k+1}\tan\frac{1}{2}(A+B)$$

Q)5) If $A+B+C = \pi$ prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

Proof

$LHS = \sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$= 2\sin C\cos(A-B) + 2\sin C\cos C$ [$\because A+B = \pi - C$. $\sin(A+B) = \sin(\pi - C) = \sin C$)

$= 2\sin C\{\cos(A-B) + \cos C\}$

$= 2\sin C(\cos(A-B) - \cos(A+B))$ [$\because \cos C = \cos[\pi - (A+B)] = -\cos(A+B)$]

$= 2\sin C(2\sin A\sin B)$

$= 4\sin A\sin B\sin C = (RHS)$

Theorem - 1

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$, $A \neq (2n+1)\dfrac{\pi}{2}$

Proof

i) According to addition theorem $\sin(A+B) = \sin A \cos B + \cos A \sin B$. Replace the angle B by A $(\sin A + A) =$

$= \sin A \cos A + \cos A \sin A$

or, $\sin 2A = 2\sin A \cos A$

(ii) According to addition theorem, $\cos(A+B) = \cos A \cos B - \sin A \sin B$. Replace the angle B by A $(\cos A + A)$

$= \cos A \cos A - \sin A \sin A$

or, $\cos 2A = \cos^2 A - \sin^2 A$

Again, by using Identity $\sin^2 A + \cos^2 A = 1$ in above form

or, $\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$

and $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - \sin^2 A$

Again by using the Identity $\sin^2 A + \cos^2 A = 1$

(iii) According to addition theorem $\tan(A+B)$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Replace the Angle B by A, $\tan(A+A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

or $\tan 2A = \dfrac{2\tan A}{1 - \cancel{8}\tan^2 A}$

Theorem - 2 :

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^2 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan A}$

Proof

(i) $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= (2\sin A \cos A)\cos A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A \cos^2 A + \sin A - 2\sin^3 A$

$= 2\sin A (1 - \sin^2 A) + \sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = \cos(3A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A \cos A)\cos A - (2\sin A \cdot \cos A)\sin A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \tan(2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$= \dfrac{\left(\dfrac{2\tan A}{1 - \tan^2 A}\right) + \tan A}{1 - \left(\dfrac{2\tan A}{1 - \tan A}\right)\tan A}$

$= \dfrac{3\tan A - \tan^2 A}{1 - 3\tan^2 A}$

Note : Replace A by A/2 in theorem-1 the following can be proved.

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

Note : Replace A by A/3, in theorem-1, the following can be derived.

(iv) $\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$

(v) $\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

(vi) $\tan\theta = \dfrac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}$

Some Solved problems :

Q.1 : prove that $\dfrac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A$.

$$\text{LHS} = \frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}}{\frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}} = \frac{\frac{\cos^2 A - \sin^2 A}{\sin A\cos A}}{\frac{\cos^2 A + \sin^2 A}{\sin A\cos A}}$$

$$= \frac{\cos^2 A - \sin^2 A}{\sin A\cos A} \times \frac{\sin A\cos A}{\cos^2 A + \sin^2 A} = \cos 2A = (\text{RHS})$$

Q.2 : prove that $\cot\frac{A}{2} = \dfrac{\sin A}{1-\cos A}$

Proof

~~takes~~ $\text{RHS} = \dfrac{\sin A}{1-\cos A} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin^2\frac{A}{2}} = \dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2}$

(RHS)

Q:3: Prove that $\cot A - \operatorname{cosec} 2A = \cot 2A$

Proof

LHS : $\cot A - \operatorname{cosec} 2A = \frac{\cos A}{\sin A} - \frac{1}{\sin 2A}$

$= \frac{\cos A}{\sin A} - \frac{1}{2\sin A \cos A} = \frac{2\cos^2 A - 1}{2\sin A \cos A}$

$= \frac{\cos^2 A}{\sin^2 A} = \cot 2A$

Q:4: Find the value of $\sin 20^\circ (3 - 4\cos^2 70^\circ)$?

$\sin 20^\circ (3 - 4\cos^2 70^\circ) = \sin 20^\circ [3 - 4\cos^2 (90^\circ - 20^\circ)]$

$= \sin 20^\circ [3 - 4\sin^2 20^\circ)$

$= 3\sin 20^\circ - 4\sin^3 20^\circ$

$= \sin(3 \times 20^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

Q:5: prove that $\cos^6 A - \sin^6 A = \cos 2A (1 - \frac{1}{4}\sin^2 2A)$

LHS $= \cos^6 A - \sin^6 A = (\cos^2 A)^3 - (\sin^2 A)^3$

$= (\cos^2 A - \sin^2 A)(\cos^4 A + \cos^2 A \sin^2 A + \sin^4 A)$

$= \cos 2A \{(\sin^2 A)^2 + (\cos^2 A)^2 + \sin^2 A \cos^2 A$

$= \cos 2A \{(\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A \cos^2 A - 2\sin^2 A \cos^2 A + \sin^2 A \cos^2 A\}$

$= \cos 2A \{(\cos^2 A + \sin^2 A)^2 - \sin^2 A \cos^2 A\}$

$= \cos 2A \{1 - (\sin A \cos A)^2\}$

$= \cos 2A \{1 - (\frac{1}{2} \cdot 2\sin A \cos A)^2\}$

$= \cos 2A (1 - \frac{1}{4}\sin^2 2A) =$ RHS

Q:6: prove that $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

LHS $= \sin 50^\circ - \sin 70^\circ + \sin 10^\circ$

$= \sin(60^\circ - 10) - \sin(60^\circ + 10^\circ) + \sin 10^\circ$

$= -2\cos 60^\circ \sin 10^\circ + \sin 10^\circ$

$= -2 \times \frac{1}{2} \sin 10^\circ + \sin 10^\circ$

$= -\sin 10^\circ + \sin 10^\circ = 0$ (RHS)

Q: 7: find the value of $2\sin 67\frac{1}{2} \cos 22\frac{1}{2}$

$2\sin(90^\circ - 22\frac{1}{2})\cos 22\frac{1}{2} = 2\cos 22\frac{1}{2}\cos 22\frac{1}{2}$ [$\because \sin(90^\circ$ ... $= \cos$ ...]

$= 2\cos^2 22\frac{1}{2}$

$= 2\cos^2 22\frac{1}{2} - 1 + 1$

$= \cos 2 \times 22\frac{1}{2} + 1$

$= \cos 45^\circ + 1 = \frac{1}{\sqrt{2}} + 1$

Q: 8: Prove that $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

we know that $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ ——— (1)

put $\theta = \frac{\pi}{4}$, $1 + \cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}$

$\Rightarrow 1 + \frac{1}{\sqrt{2}} = 2\cos^2\frac{\pi}{8}$

$\Rightarrow 4\cos^2\frac{\pi}{8} = 2(1+\frac{1}{\sqrt{2}}) = 2+\sqrt{2}$

$\Rightarrow 2\cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$

Put $\theta = \frac{\pi}{8}$ in eqn (1) we get

$2\cos^2\frac{\pi}{16} = 1 + \cos\frac{\pi}{8}$

$\Rightarrow 4\cos^2\frac{\pi}{16} = 2 + 2\cos\frac{\pi}{8}$

$\Rightarrow \left(2\cos\frac{\pi}{16}\right)^2 = 2 + \sqrt{2+\sqrt{2}}$

$\Rightarrow 2\cos\frac{\pi}{16} = \sqrt{2+\sqrt{2+\sqrt{2}}}$ (proved)

Function	Domain (D)	Range (R)
$\sin^{-1}x$	$-1 \le x \le 1$ or $[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}x$	$-1 \le x \le 1$ or $[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	$[-\infty, -1] \cup [1, \infty]$	$(0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$
$\csc^{-1}x$	$[-\infty, -1] \cup [1, \infty]$	$[-\frac{\pi}{2}, 0] \cup [0, \frac{\pi}{2}]$

Properties of inverse trignometric functions.

1) Self adjusting property:

(i) $\sin^{-1}(\sin\theta) = \theta$

(ii) $\cos^{-1}(\cos\theta) = \theta$

(iii) $\tan^{-1}(\tan\theta) = \theta$

Proof

1) Let $\sin\theta = x$, then $\theta = \sin^{-1}x$

$\therefore \sin^{-1}(\sin\theta) = \sin^{-1}x = \theta$ (proved)

Similarly, proofs of (ii) & (iii) can be completed.

2) Reciprocal property:

(i) $\operatorname{cosec}^{-1}\frac{1}{x} = \sin^{-1}x$

(ii) $\sec^{-1}\frac{1}{x} = \cos^{-1}x$

(iii) $\cot^{-1}\frac{1}{x} = \tan^{-1}x$

Proof

1) Let $\sin^{-1}x = \theta$, $\Rightarrow x = \sin\theta$

Then, $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x}$, $\Rightarrow \theta = \operatorname{cosec}^{-1}\frac{1}{x}$

Hence $\sin^{-1}x$ and $\theta = \operatorname{cosec}^{-1}\frac{1}{x}$

Therefore, $\sin^{-1}x$ $\operatorname{cosec}^{-1}\frac{1}{x}$

$\therefore \sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$ (proved)

Similarly (ii) & (iii) can be proved.

3) Conversion property

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

Proof:

1). Let $\theta = \sin^{-1}x$, so that $\sin x = \theta$

Now $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e $\theta = \cos^{-1}\sqrt{1-x^2}$

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$ or, $\theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

thus $\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ (proved

Similar (2) & (3) can also be proved.

4) Theorem 1 :

1) $\sin^{-1}x + \cos^{-1}x = \pi/2$

2) $\tan^{-1}x + \cot^{-1}x = \pi/2$

3) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$

Proof :

(i) Let $\sin^{-1}x = \theta$

$\Rightarrow x = \sin\theta = \cos(\pi/2 - \theta)$

$\Rightarrow \cos^{-1}x = \pi/2 - \theta = \pi/2 - \sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = \pi/2$ (proved)

Similarly (2) & (3) can be proved.

5) Therom -2

If $xy < 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof :

Let $\tan^{-1}x = \theta$ and $\tan^{-1}y = \theta_2$ then

$x = \tan\theta_1$ & $y = \tan\theta_2$

Now, $\tan(\theta_1+\theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$

$\Rightarrow \tan(\theta_1+\theta_2) = \frac{x+y}{1-xy}$

$\Rightarrow (\theta_1+\theta_2) = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(proved)

6) **Theorem - 3**

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{xy}{1-xy}\right)$

Proof :

Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$

then $x = \tan\theta_1$ and $y = \tan\theta_2$

Now $\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1+\tan\theta_1 + \tan\theta_2}$

$\Rightarrow \tan(\theta_1 - \theta_2) = \frac{xy}{1+xy}$

$\Rightarrow (\theta_1 - \theta_2) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\Rightarrow \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{xy}{1+xy}\right)$ (proved)

Note

1) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left\{\frac{x+y+z-xyz}{1-xy-yz-zx}\right.$

2) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$ if $|x|<1$

$= \sin^{-1}\frac{2x}{1+x^2}$ if $|x| \leq 1$

$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ if $|x| \geq 0$

7) **Theorem - 4**

i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) $2\cos^{-1}x = \cos^{-1}\left[2x^2-1\right]$

Proof

1) $\sin^{-1}x = \theta$, then $x = \sin\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2x\sqrt{1-x^2}$

$\Rightarrow 2\theta = \sin^{-1} 2x\sqrt{1-x^2}$

$\Rightarrow 2\sin^{-1}x = \sin^{-1} 2x\sqrt{1-x^2}$ (proved)

(ii) Let $\cos^{-1}x = \theta$ then $x = \cos\theta$

$\therefore \cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos 2\theta = 2x^2 - 1$

$\Rightarrow 2\theta = \cos^{-1}(2x^2-1)$

$\Rightarrow 2\cos^{-1}x = \cos^{-1}(2x^2-1)$ proved

8) Theorem: 5

i) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(ii) $3\cos^{-1}x = \cos^{-1}(3x-4x^3)$

(iii) $3\tan^{-1}x = \tan^{-1}\left\{\dfrac{3x-x^3}{1-3x^3}\right.$

Proof:

1) Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$

we know that

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \sin 3\theta = 3x - 4x^3$

$\Rightarrow 3\theta = \sin^{-1}(3x-4x^3)$

$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

Similarly (2) & (3) can also be proved

9) Theorem - 6

1) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

2) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

3) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2}\ y\sqrt{1-x^2})$

4) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$

Proof :

i) Let $\sin^{-1}x = \theta$, and $\sin^{-1}y = \theta_2$

Then, $x = \sin\theta$, & $y = \sin\theta_2$

$\therefore \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

$= \sin\theta_1 \sqrt{1-\sin\theta_2} + \sqrt{1-\sin^2\theta_1}\sin\theta_2$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

Similarly, others can also be proved.

10) Theorem - 7

(i) $\sin^{-1}(-x) = -\sin^{-1}x$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$

Proof

(i) Let $-x = \sin\theta \Rightarrow \theta = \sin^{-1}(-x)$

Since, $-x = \sin\theta$

$\Rightarrow x = -\sin\theta = \sin(-\theta)$

$\Rightarrow -\theta = \sin^{-1}x$

$\Rightarrow \theta = -\sin^{-1}x$

From eqn (1) & (2) $\sin^{-1}(-x) = -\sin^{-1}x$. (proved)

(ii) Let $-x = \cos\theta$, $\rightarrow \theta = \cos^{-1}(-x)$

since $-x = \cos\theta$

$\Rightarrow x = -\cos\theta = \cos(\pi - \theta)$

$\Rightarrow \pi - \theta = \cos^{-1}x$

$\Rightarrow \theta = \pi - \cos^{-1}x$

From eqn (3) & (4) $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

(iii). Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}(-x)$

since $-x = \tan\theta$

$\Rightarrow x = -\tan\theta = \tan(-\theta)$

$\Rightarrow -\theta = \tan^{-1}x$

$\Rightarrow \theta = -\tan^{-1}x$

from eqn (5) & (6) $\tan^{-1}(-x) = -\tan^{-1}x$.

Some solved Problems:

Q) 1) find the value of $\cos\tan^{-1}\cot\cos^{-1}\sqrt{3}/2$

$\cos\tan^{-1}\cos\cos^{-1}\sqrt{3}/2$

$= \cos\tan^{-1}\cot\cos^{-1}\cos\pi/6$

$= \cos\tan^{-1}\cot\pi/6$

$= \cos\tan^{-1}\tan(\pi/2 - \pi/6)$

$= \cos(\pi/2 - \pi/6) = \sin\pi/6 = 1/2$

Q2) prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

Proof

LHS. $=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right.$

$=\tan^{-1}\left(\frac{5/6}{1-1/6}\right)=\tan^{-1}\left(\frac{5/6}{5/6}\right)=\tan^{-1}(1)=\frac{\pi}{4}$ (RHS)

Q3) Prove that $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\cos^{-1}\frac{16}{65}$

Proof

$=\sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right]\left[\begin{array}{l}\because \sin^{-1}x+\sin^{-1}y \\ \sin^{-1}x\sqrt{1-y^2}+y\sqrt{1-x}\end{array}\right]$

$=\sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right.$ $\sqrt{1-\frac{16}{25}}$

$=\sin^{-1}\left[\frac{4}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{3}{5}\right]$

$=\sin^{-1}\left[\frac{63}{65}\right]=\cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2}=\cos^{-1}\frac{16}{25}=$ RHS.

Q4) Prove that $2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}$

Proof

We know that $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\therefore 2\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)=\tan^{-1}\left(\frac{2/3}{8/9}\right)$

$=\tan^{-1}\frac{3}{4}$ (RHS)

Q)5) show that $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

PROOF

LHS $= \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16}$

$= \tan^{-1}\left(\frac{12/13}{\sqrt{1-(12/13)^2}}\right) + \tan^{-1}\left(\frac{\sqrt{1-(4/5)^2}}{4/5}\right) + \tan^{-1}\frac{63}{16}$

$\left[\because \sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right.$
$\left.\cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right]$

$= \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\frac{63}{16}$

$= \tan^{-1}\left(\frac{\frac{15}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right) + \tan^{-1}\frac{63}{16}$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left(-\frac{63}{16}\right) + \tan^{-1}\frac{63}{16}$

$= \tan^{-1}0 = \pi = \text{RHS}$

Q)6) Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

LHS $= 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-(1/2)^2} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1}\frac{(31/21)}{(17/21)}$

$= \tan^{-1}\frac{31}{17} = \text{RHS}.$

Q7) prove that $\cot^{-1} 9 + \text{cosec}^{-1} \sqrt{41}/4 = \pi/4$

$LHS = \cot^{-1} 9 + \text{cosec}^{-1} = \frac{\sqrt{41}}{4}$

$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \left(\because \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \tan^{-1}\frac{4}{5}\right)$

$= \tan^{-1} \dfrac{\frac{1}{9} + \frac{5}{4}}{1 - (\frac{1}{9})(\frac{4}{5})}$

$= \tan^{-1} \dfrac{41/45}{41/45} = \tan^{-1} 1 = \pi/4 \, (RHS)$

Q)8) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then prove that $x^2 + y^2 + z^2 + 2xyz$.

Given $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \pi \cdot \cos^{-1} z$

$\Rightarrow (xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi \cdot \cos^{-1} z)$

$\Rightarrow (xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos(\cos^{-1} z)$

$\Rightarrow (xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow (xy + z) = (\sqrt{1-x^2}\sqrt{1-y^2})$

Squaring both sides.

$\Rightarrow (xy + z)^2 = (\sqrt{1-x^2}\sqrt{1-y^2})^2$

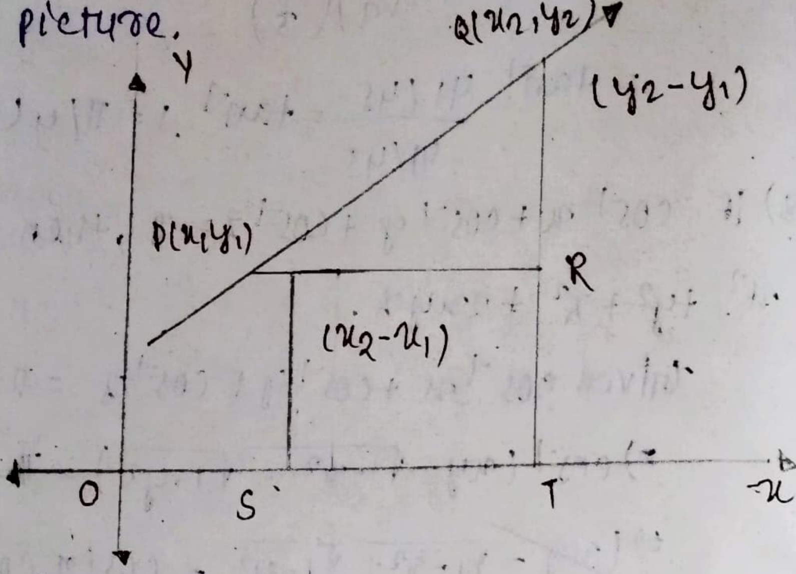
$\Rightarrow (xy)^2 + z^2 + 2xyz = (1-x^2)(1-y^2)$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

Distance formula:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in the co-ordinate picture.



ΔPQR is a right angle tringle.

By pythagoras theorem

$$PQ^2 = PR^2 + QR^2$$

or, $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is the required distance between two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$

or, $PQ = \sqrt{(\text{Difference of abscissa})^2 + (\text{Difference of ordinates})^2}$

Distance between a point from the origin:

Distance of a point $P(x, y)$ from the origin $O(0,0)$ is

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Some Solved Problems:

Q-1: Find the distance between the points $P(1,2)$ and $Q(2,-3)$.

Sol:

The distance between the points $P(1,2)$ and $Q(2,-3)$ is:

$$|PQ| = \sqrt{(2-1)^2 + (-3-2)^2} = \sqrt{1+25} = \sqrt{26} \text{ units.}$$

Q:-2: If the distance between the points $(3, a)$ and $(6, 1)$ is 5. Find 'a'.

Sol: Given the distance between the points $(3, a)$ and $(6, 1) = 5$,

using distance formula

$$\sqrt{(6-3)^2 + (1-a)^2} = 5$$

$$\text{or, } 9 + (1-a)^2 = 25$$

$$(1-a)^2 = 16$$

$$1-a = \pm 4$$

$$a = 5, -3$$

Q:3: If $O(0,0)$ $A(1,0)$ $B(1,1)$ are the verties of the tringle, what type of tringle is ΔOAB?

Sol:

Given $O(0,0)$, $A(1,0)$, $B(1,1)$ are the vertices of the traingle ΔOAB.

using distance formula,

$$|OA| = \sqrt{(1-0)^2+(0-0)^2} = 1$$

$$|OB| = \sqrt{(1-0)^2+(1-0)^2} = \sqrt{2}$$

$$|AB| = \sqrt{(1-1)^2+(1-0)^2} = 1$$

$OA = AB$ and $|OA|^2 + |AB|^2 = |AB|^2$

therefore

Hence, ΔOAB is a right angle isosceles traingle.

Internal divison formula:

$$\text{The co-ordinates} = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

External division formula:

$$\text{The co-ordinates} = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

Mid point formula

$$\text{The co-ordinates} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Some Solved Problems :

Q-1 : Find the co-ordinates of the point which divides the line joining the points P(1,2) and Q (3,4) in the ratio 2:1 internally.

Sol :

Given P(1,2) Q(3,4) be two points.

Let R be the point which divides the PQ internal in the ratio 2:1,

Using internal divison formula ; the co-ordinates of point R are

$$\left(\frac{(2\times 3)+(1\times 1)}{2+1}, \frac{(2\times 4)+(1\times 2)}{2+1}\right) = \left(\frac{7}{3}, \frac{10}{3}\right)$$

Q-2: Find the co-ordinates of the point which divides the line joining the points P(2,3) and Q (-3,1) in the ratio 3:2 externally.

Sol :

Let R be the point which devides PQ, joining P(2,3) and Q(-3,1), externally in the ratio 3:1

using external division formula,

The co-ordinate of point R are

$$\left(\frac{(3\times -3)-(2\times 2}{3-2}, \frac{(3\times 1)-(2\times 3}{3-2}\right) = (-13, -3)$$

Q-3: Find midpoint of the line joining P(2,3) and Q(4,5)

Sol:

Let R be the midpoint of the line joining P(2,3) and Q(4,5)

Using the mid-point formula, the co-ordinates of R are $\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = (3,4)$

Q:4:- In what ratio does the point (-1,-1) divide the line segment joining the points (4,4) and (7,7)

Sol:

Let the point C(-1,-1) divides the line segment joining the point A(4,4) and B(7,7) in the ratio K:1

Then the co-ordinates of point C are

$$\left(\frac{7K+4}{K+1}, \frac{7K+4}{K+1}\right)$$

Therefore $\frac{7K+4}{K+1} = -1$

$7K+4 = -K-1$

$8K = -5$

$K = \frac{-5}{8}$

Hence the point C divides AB externally in the ratio 5:8.

Q-5: In what ratio does the x-axis divide the line segment joining the points (2, -3) and (5, 6)?

Sol:

The co-ordinates of the point which divides the line segment joining the points (2, -3) and (5, 6) internally in the ratio k:1 are $\left(\frac{5k+2}{k+1}, \frac{6k-3}{k+1}\right)$

As, this point lies on x-axis where y-co-ordinate of every point is zero.

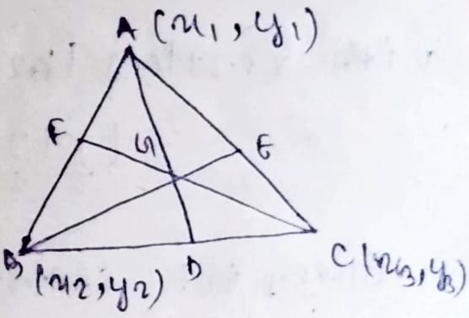
Therefore $\frac{6k-3}{k+1} = 0$ or, $6k-3=0$ or, $k=\frac{1}{2}$

Hence, the required ratio is 1:2

Centroid of a tringle

The co-ordinates are

$$=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$



Some solved Problem :

Q1: Find the co-ordinates of centroid of the tringle whose vertics are (0,6), (8,12) and (8,0).

Sol:

We know that, the co-ordinates of the centroid of a tringle with vertics (x_1, y_1), (x_2, y_2) and (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Therefore the co-ordinates of the centroid of the triangle with vertices $(0,6)$ $(8,12)$ and $(8,0)$

are $\left(\frac{0+8+8}{3}, 6+12+0 \right.$

Q-2: Two vertices of a triangles are $(1,2)$ $(3,5)$ and its centroid is at the origin. Find the co-ordinates of the third vertex.

Sol:

Let the co-ordinates of the third vertex of the triangle be (x, y).

So, the co-ordinates of the centroid of a triangle with vertices $(1,2)$, $(3,5)$ and (x,y) are given by

$$\left(\frac{1+3+x}{3}, \frac{2+5+y}{3} \right)$$

Given the centroid is at the origin $(0,0)$

Therefore, $\frac{4+x}{3} = 0$ and $\frac{7+y}{3} = 0$

$\therefore$ $x = -4$ and $y = -7$

Hence the co-ordinates of the third vertex are $(-4,-7)$.

Area of a tringle :

$$= \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

<u>Co - linearity of three points</u> :

If the point $A(x_1, y_1)$ $B(x_2, y_2)$ and $C(x_3, y_3)$ are colliner if they lie on a straight line i.e area of the $\Delta ABC = 0$

i.e $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

<u>Some Solved Problems</u> :

Q:1 - Find the area of tringle whose vertices are $A(4, 4)$ $B(3, -2)$ and $C(-3, 16)$.

Sol :

$$\text{Area of the } \Delta ABC = \frac{1}{2} \begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[4(-2-16) - 4(3+3) + 1(48-6) \right]$$

$$= \frac{1}{2} \left[-72 - 24 + 42 \right] = \frac{1}{2}(-54) = -27$$

$\therefore$ Area of the tringle $= |-27| = 27$ square

Q:2- Find the value of 'a' so that area of the triangle having vertices A(0,0) B(1,0) and C(0,a) is 10 units

Sol:

Given Area of $\Delta ABC = 10$ units

$$\text{or } \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = 10$$

$$1(a-0) = 20$$

$$a = 20 \text{ units.}$$

Q:3- Find the value of a so that A(1,4) B(2,7) and C(3,a) are collinear.

Sol:

Given A(1,4) B(2,7) and C(3,a) are collinear.

Therefore Area of $\Delta ABC = 0$

$$\frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 3 & a & 1 \end{vmatrix} = 0$$

$$1(7-a) - 4(2-3) + 1(2a-21) = 0$$

$$7 - a + 4 + 2a - 21 = 0$$

$$a = 10$$

Slope (Gradient) of a line :

The tangnet of the angle made by a line with the positive direction of the x-axis in anti clockwise sense is called slope or gradient of the line.

Hence, $m = \tan\theta$

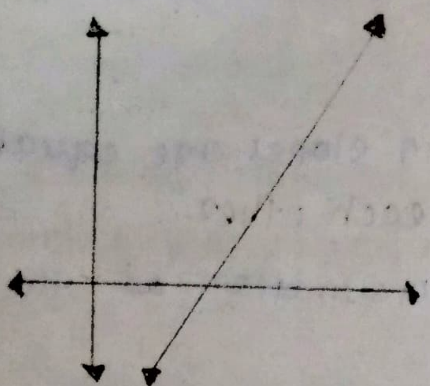


fig 3.9

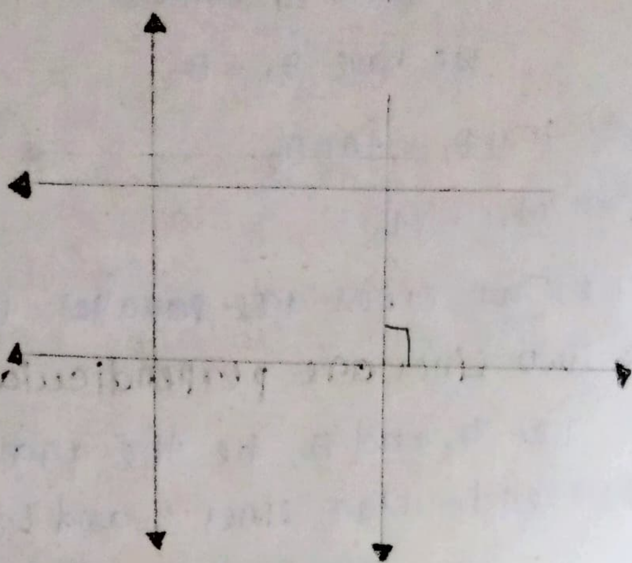


fig 3.10

Note : 1. In (fig 3.10) L_1 is the line parallel to x-axis so $\theta = 0^\circ \Rightarrow m = \tan 0^\circ = 0$ So, slope of the line parallel to x-axis is zero.

Note : 2. The line L_2 is perpendicular to x-axis or parallel to y-axis, so $\theta = 90^\circ \Rightarrow m = \tan 90^\circ$ not defined

Slope of a line-joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

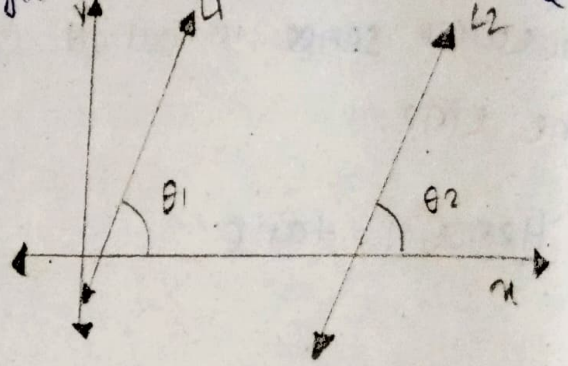
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line.

$$m = \tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Conditions of parallelsim and perpendicularity:

1. Two lines L_1 and L_2 are parallel.:

Let θ_1 and θ_2 be the angle of Inclination of the parallel lines L_1 and L_2



we have $\theta_1 = \theta_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

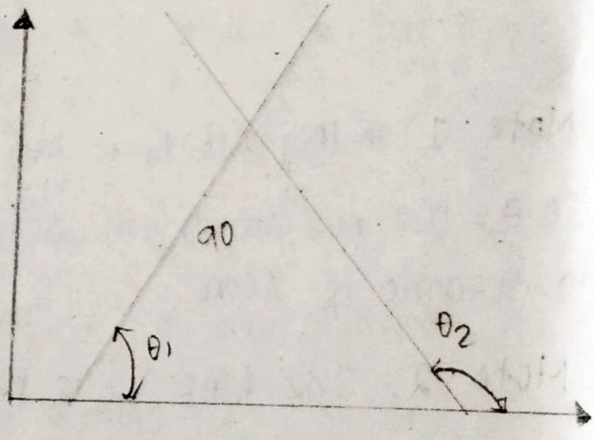
$\Rightarrow M_1 = M_2$

i.e Two lines are parallel if their slopes are equal.

2. Two lines are perpendicular to each other.

Let θ_1 and θ_2 be the angle of inclination of the perpendicular lines L_1 and L_2

we have $\theta_1 = \theta_2$



$\theta_2 = 90 + \theta_1$

$\Rightarrow \tan\theta_2 = \tan(90+\theta_1)$

$\Rightarrow M_2 = \cot\theta_1 = -\frac{1}{\tan\theta_1}$

$\Rightarrow M_2 = \frac{-1}{M_1}$

$\Rightarrow M_1 M_2 = -1$

i.e. two lines are perpendicular if their product is equal to -1.

Some Solved Problems:

Q1: Find slope of a line joining P(2,3) and Q(1,4).

Sol: Slope of the line joining P(2,3) and Q(1,4)

$= M = \frac{y_2-y_1}{x_2-x_1} = \frac{4-3}{1-2} = \frac{1}{-1} = -1$

Q:2- Find slope of the line perpendicular to a line joining $P(1,2)$ and $Q(3,5)$.

Sol:

Slope of the line PQ joining $P(1,2)$ and $Q(3,5)$

$$= \frac{y_2-y_1}{x_2-x_1} = \frac{5-2}{3-1} = \frac{3}{2}$$

So, slope of the line perpendicular to PQ

$$= \frac{-1}{3/2} = -2/3$$

(Since product of their slopes is -1)

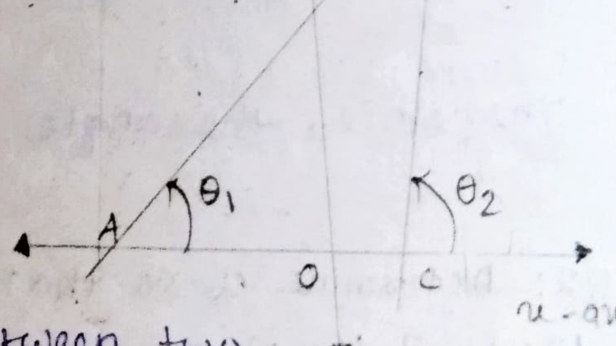
Q:3- Find slope of the line parallel to the line joining $P(1,4)$ and $Q(2,6)$

Sol:

Slope of the line PQ, joining $P(1,4)$ and $Q(2,6)$ =

$$\frac{6-4}{2-1} = \frac{2}{1} = 2$$

So, slope of the line parallel to PQ $= 2$ (since slopes of parallel lines are equal).



Let θ be the angle between two straight lines with slopes m_1 and m_2 i.e

$m_1 = \tan\theta_1$, and $m_2 = \tan\theta_2$

where θ_1 and θ_2 are the angle of inclination of two lines.

$$\theta + \theta_1 = \theta_2$$

or, $\tan\theta = \tan(\theta_2 - \theta_1)$

or, $\tan\theta = \dfrac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2 \tan\theta_1} = \dfrac{M_2 - M_1}{1+M_2 M_1}$

$$\tan\theta = \pm \frac{M_2 - M_1}{1+M_1 M_2}$$

Some solved problems :-

Q-1: If $A(-2,1)$ $B(2,3)$ and $C(-2,-4)$ are three points find the angle between BA and BC.

Sol: Let M_1 and m_2 be the slopes of BA and BC respectively

$\therefore M_1 = \dfrac{3-1}{2+2} = \dfrac{1}{2}$ and $m_1 = \dfrac{-4-3}{-2-2} = \dfrac{7}{4}$

Let θ be the angle between BA and BC.

$\therefore \tan\theta = \left|\dfrac{M_2 - M_1}{1+M_1 M_2}\right| = \left|\dfrac{\frac{7}{4} - \frac{1}{2}}{1+\frac{1}{2}\cdot\frac{7}{4}}\right| = \dfrac{5/4}{15/8} = \dfrac{2}{3}$

Therefore, the angle between BA and BC $= \theta = \tan^{-1}\left(\frac{2}{3}\right)$

Q2: Determine x so that the line passing through $(3,4)$ and $(x,5)$ makes 135° angle with the positive direction of x-axis.

Sol: The slope of the line passing through (3,4) and (x,5) $= \frac{5-4}{x-3} = \frac{1}{x-3}$

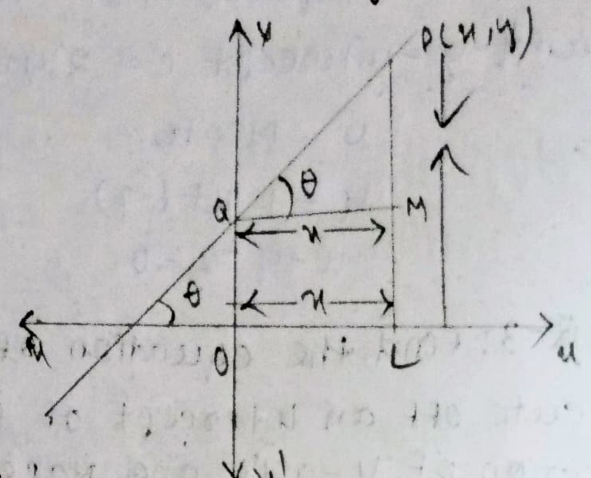
Again, the line makes 135° angle with the positive direction of x-axis,

So its slope = tan 135° = −1

Therefore $\frac{1}{x-3} = -1$ or $x = 2$

Different forms of equation of a straigh line.

1. Slope - Intercept form :



$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$

$m = \frac{y-c}{x}$

$y = mx + c$, is the required equation of the line.

Some solved problem :-

Q-1: Find equation of the line which has slope 2 and y intercept 3.

Sol:

Given slope $= m = 2$ and y - intercept, $c = 3$

Using slope - intercept form

Equation of straight line with slope $m = 1$ and y-intercept $c = -2$ is given by

$y = mx + c$

$y = 2x + 3$

$2x - y + 3 = 0$

Q-2: Find the equation of a line with slop 1 and cutting off an intercept 2 units on the negative direction of y axis.

Sol:

Let m be the slope and c be the y-intercept of the required line.

Given $m = 1$ and $c = -2$

$\therefore$ The equation of the line with slope $m = 1$ and y-intercept $c = -2$ given by

$y = mx + c$

$y = 1.x + (-2)$

$x - y - 2 = 0$

Q-3: Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of y-axis and makes an angle of 120° with positive direction of x-axis.

Sol:

Here slope, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$

and y - Intercept $c = -5$

using slope - intercept form $y = mx + c$

Therefore, the equation of the required line is

$y = -\sqrt{3}x - 5$

$= \sqrt{3}x + y + 5 = 0$

2. One point - slope form:

Let the line with slope m passe through $Q(x_1, y_1)$

Let $P(x, y)$ be any point on the line.

Then slope of the line is given by $m = \frac{y - y_1}{x - x_1}$

Therefore $y - y_1 = m(x - x_1)$ is the equation of required line.

Some Solved Problems :-

Q-1: Find equation of the line which passes through (1,2) and slope 2.

Sol:

using one point - slope form,
equation of the line passes through $(x_1, y_1) = (1,2)$ and slope $m = 2$ is given by

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 2(x - 1)$$
$$2x - y = 0$$

Q-2: Determine the equation of line through the point (4,-5) and parallel to x-axis.

Sol:

Since the line is parallel to x-axis, slop, m = 0
using point - slope form,
equation of the line passes through $(x_1, y_1) = (4 - 5)$ and slope $m = 0$ is given by

$$y - y_1 = m(x - x_1)$$
$$y + 5 = 0(x - 4)$$
$$y + 5 = 0$$

Q-3: Find equation of the line which bisects the line segment joining P(1, 2) and Q(3, 4) at right angle.

Sol:

Let R be the Mid-point of the line joining

P(1, 2) and Q(3, 4).

So, co-ordinates of R are $R\left(\frac{1+3}{2}, \frac{2+4}{2}\right) = R(2, 3)$

Now, slope of PQ $= M_{PQ} = \frac{4-2}{3-1} = \frac{2}{2} = 1$

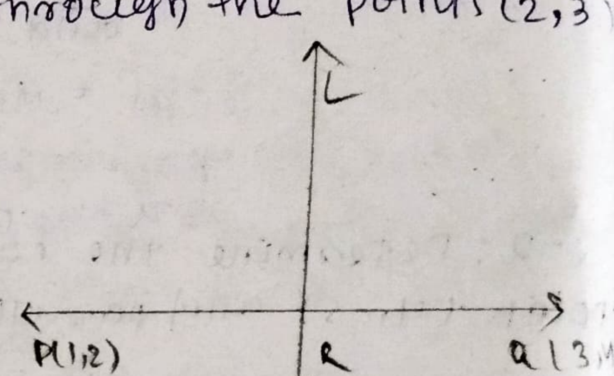
The line LR which passes through the points (2, 3) and slope -1 is

$y - y_1 = m(x - x_1)$

$y - 3 = -1(x-2)$

$y - 3 = -x + 2$

$x + y - 5 = 0$



3. <u>Two - point form</u>:

Let m be the slope of a line passing through two points (x_1, y_1) and (x_2, y_2)

$\therefore$ slope $m = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore$ the equation of the required line is

$y - y_1 = m(x - x_1)$ (one point - slope form)

putting $m = \frac{y_2 - y_1}{x_2 - x_1}$ in above eqution, we get

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of required line.

Some Solved Problems:

Q-1: Find equation of the line which passes through two points P(1,2) and Q(3,4).

Sol:

Let m be the slope of the line PQ joining the points P(1,2) and Q(3,4).

$\therefore$ slope $= m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = 1$.

Here, $x_1 = 1, y_1 = 2$ and $x_2 = 3$ $y_2 = 4$

using two-point form

Equation of the required line is

$$y - y_1 = m(x - x_1),$$

$$y - 2 = 1(x-1)$$

$$x - y + 1 = 0$$

Q-2: Prove that the points (5,1), (1,-1) and (11,4) are collinear, find the equation of the line on which these points lie.

Proof:

Let the given points be A(5,1) B(1,-1) and C(11,4). Then the equation of the line passing through A(5,1) and B(1,-1) is

$$y - 1 = \frac{-1-1}{1-5}(x-5)$$

$$y - 1 = \frac{1}{2}(x-5)$$

$$x - 2y - 3 = 0$$

Put $x = 11$ and $y = 4$ in the above equation we get

$11 - 2\times4 - 3 = 0$,

Clearly, the point C(11,4) satisfies the equation

$$x - 2y - 3 = 0$$

Hence, the given points A, B and C lie on the same straight line and whose equation is $x-2y-3=0$

4. Intercept form:

Let AB be a straight line cutting the x-axis and y-axis at $A(a,0)$ and $B(0,b)$ respectively (Fig 3.16).

Let x-intercept $= OA = a$ and
Let y-intercept $= OB = b$

Therefore, using two-point form
The equation of the required straight line passing through $A(a,0)$ and $(0,b)$

By two point form, its equation is given by

$$(y-0) = \frac{b-0}{0-a}(x-a)$$

$$y = -\frac{b}{a}(x-a)$$

$$\therefore bx + ay = ab$$

Dividing both sides by ab

or, $\frac{x}{a} + \frac{y}{b} = 1$. is the equation of line in intercept form.

Some solved problem:-

Q-1: Find equation of the line which has x-intercept is 2 and y-intercept equal to 3.

Sol: Given x-intercept $= a = 2$ and y-intercept $= b = 3$
using intercept form
Equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{2} + \frac{y}{3} = 1$$

$$3x + 2y - 6 = 0$$

Q2: Find the equation of the straight line which makes equal intercepts on the axis and passes through the point $(2,3)$.

Sol :-

Let the equation of the line with intercept a and b is $\frac{x}{a}+\frac{y}{b}=1$

Since it makes equal intercepts a and b on the co-ordinate axis, then $a=b$

$\therefore$ equation of the line is $\frac{x}{a}+\frac{y}{a}=1$ or $x+y=0$

The line $x+y=a$ passes throug the point $(2,3)$

So, $2+3=a$ or $a=5$

Hence, the equation of the required line is

$$\frac{x}{5}+\frac{y}{5}=1 \text{ or, } x+y=5$$

Q-3: Find the equation of the straight line which passes through the point $(3,4)$ and the sum of the intercepts on the axis is 14.

Sol :

Let the equation of the line with intercepts a and b $\frac{x}{a}+\frac{y}{b}=1$

Given, the line passes through the point $(3,4)$.

$\therefore \frac{3}{a}+\frac{4}{a}=1$ or $3b+4a=ab$

Also given that, sum of intercepts $=14$ i.e $a+b=14$

solving eqn (2) and (3), we have,

$$3(14-a)+4a=a(14-a)$$

$$a^2-13a+42=0$$

$$(a-6)(a-7)=0$$

$$a=6 \text{ and } 7$$

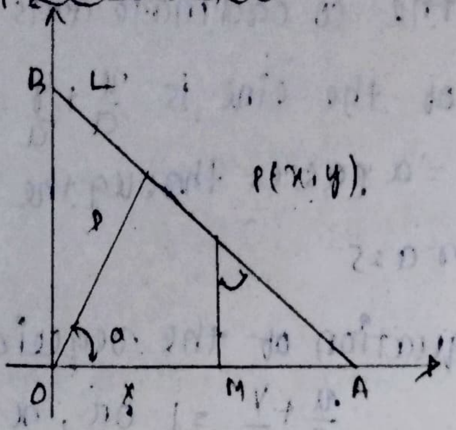
For $a=6$, the value of $b=8$ and for $a=7$ the value of $b=7$.

putting the values of a and b in eqn (1), we get,

$$\frac{x}{6}+\frac{y}{8}=1 \text{ and } \frac{x}{7}+\frac{y}{7}=1$$

or $4x+3y=24$ and $x+y=7$ are the equations of the required lines.

Normal form / perpendicular form :-



$$p = x\cos\alpha + y\sin\alpha$$

is the eqn of required line and known as normal or ⊥ form

Some solved problems :-

Q-1: Find eqn of the line which is at a distance 2 form the origin and the perpendicular from the origin to the line makes an angle of 30° with +ve direction of x-axis.

Soln

The required line is 2 unit distance from the origin i.e the perpendicular distance from the origin to the required line $p=2$

Let α be the angle make by the perpendicular from the origin with positive x-axis and given that $\alpha=30°$

using normal form, the equation of the required line is,

$x\cos\alpha + y\sin\alpha$

$x\cos 30 + y\sin 30 = 2$

$x\frac{\sqrt{3}}{2} + y\frac{1}{2} = 2$

$\sqrt{3} + y - 4 = 0$

Transformation of general equation in different standard form.

The general equation of a straight line is $Ax + By + c = 0$ which can be transformed to various standard forms as discussed below.

To transform $ax + by + c = 0$ in the slope intercept form $(y = mx + c)$

we have $Ax + By + c = 0$

$By = -Ax - C$

$y = \left(\frac{-A}{B}\right)x + \left(-\frac{C}{B}\right)$, which is of the form of $y = mx + c$, where $m = -\frac{A}{B}$ $c = -\frac{C}{B}$;

Thus, for the straight line $Ax + By + C = 0$

Slope $= m = -\frac{A}{B} = -\frac{\text{coficient of } x}{\text{coficient of } y}$

and intercept on y-axis $= \frac{C}{B} = -\frac{\text{constant term}}{\text{coficient of } y}$

To transform $Ax + By + c = 0$ in intercept $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

We have $Ax + By + C = 0$

$Ax + By = -C$

$$\frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\frac{x}{\left(\frac{C}{A}\right)} + \frac{y}{\left(\frac{C}{B}\right)} = 1$$ which is of the form $\frac{x}{a} + \frac{y}{b} = 1$

Thus, for the straight line $Ax + By + C = 0$

intercept on x-axis $= -\frac{C}{A} = \frac{\text{constant term}}{\text{coficient of } x}$

intercept on y-axis $= -\frac{C}{B} = \frac{\text{constant term}}{\text{coficient of } y}$

To transform $Ax + By + C = 0$ in the normal form ($x\cos\alpha + y\sin\alpha = p$)

we have $Ax + By + C = 0$

Let $x\cos\alpha + y\sin\alpha - p = 0$

Be the normal form of $Ax + By + C = 0$

If the equations (1) and (2) represent the same straight line.

Therefore, $\frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$

$\cos\alpha = -\frac{Ap}{C}$ and $\sin\alpha = \frac{Bp}{C}$

$$\cos^2\alpha + \sin^2\alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$$

$$1 = \frac{p^2}{C^2}(A^2 + B^2)$$

$$p = \pm \frac{C}{\sqrt{A^2+B^2}}$$

But p denots the length of the perpendicular from the origin to the line and is always positive

Therefore, $p = \frac{C}{\sqrt{A^2+B^2}}$

Putting the value of p in (3) we get $\cos\alpha = -\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = -\frac{B}{\sqrt{A^2+B^2}}$

Therefore, eqⁿ (2) takes of the form $-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y - \frac{C}{\sqrt{A^2+B^2}} = 0$

$$-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

which is required normal form of the line $Ax+By+C=0$

Some solved questions :-

Q1: Transform the equation of the line $\sqrt{3}x+y-8=0$ to (i) slope intercept form and find its slope and y-intercept.

(ii) intercept form and find intercept on the co-ordinate axes.

(iii) normal form and find the inclination of the perpendicular segement from the origin on the line with the axis and its length.

Sol :-

(i) We have $\sqrt{3}x+y-8=0$ or $y = -\sqrt{3}x+8$.

This is the slope intercept form of the given line. Therefore, slope $= -\sqrt{3}$ and y - intercept $= 8$

(ii) we have $\sqrt{3}x+y-8=0$ $\quad \frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$

This is the intercept form of the given line.
Therefore, x- intercept $= \frac{8}{\sqrt{3}}$, y - intercept = 8.

(III) we have $\sqrt{3}x + y - 8 = 0$

$$\sqrt{3}x + y = 8$$

$$\frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + 1^2}}x + \frac{1}{\sqrt{(\sqrt{3})^2 + 1^2}}y = \frac{8}{\sqrt{(\sqrt{3})^2 + 1^2}}$$

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4$$

This is the normal form of the given line.
Therefore $\cos\alpha = \frac{\sqrt{3}}{2}$ $\sin\alpha = \frac{1}{2}$ and $p = 4$
since $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant and is equal to $\alpha = \frac{\pi}{6}$,
Hence $\alpha = \frac{\pi}{6}$ and $p = 4$

Q=2 Reduce the lines $3x - 4y + 4 = 0$ and $4x - 3y + 12 = 0$ to the normal form and hence determine which line is nearer to the origin.

Sol :-

The equation of the given line is $3x - 4y + 4 = 0$

$$-3x + 4y = 4$$

$$-\frac{3x}{\sqrt{(-3)^2 + 4^2}} + \frac{4y}{\sqrt{(-3)^2 + 4^2}} = \frac{4}{\sqrt{(-3)^2 + 4^2}}$$

$$-\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$$

This is the normal form of $3x - 4y + 4 = 0$ and the length of the perpendicular from the origin to it is $P_1 = \frac{4}{5}$

Again, the equation of second line be $4x - 3y + 12 = 0$

$-4x + 3y = 12$

$$\frac{-4x}{\sqrt{(4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(4)^2+3^2}}$$

$$-\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$$

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is $P_2 = \frac{12}{5}$

Clearly $P_2 > P_1$ therefore the line $3x - 4y + 7 = 0$ is nearer to the origin.

Q-3: Find the equation of a line with slope 2, and the length of perpendicular from the origin equal to $\sqrt{5}$.

Sol:

Let c be the intercept on y-axis.
Then the equation of the lines is $y = 2x + c$

$-2x + y = c$

$$-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{1}{\sqrt{(2)^2+1^2}}y = \frac{c}{\sqrt{(-2)^2+1^2}}$$

(Dividing both side by $\sqrt{(-2)^2+1^2}$)

$$-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{c}{\sqrt{5}}$$ which is the normal form of (1)

Therefore RHS Denotes the length of the perpendicular from the origin. But the length of the perpendicular from the origin is $\sqrt{5}$

$$-4x + 3y = 12$$

$$-\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$$

$$-\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$$

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is $P_2 = \frac{12}{5}$.

Clearly $P_2 > P_1$ therefore the line $3x - 4y + 4 = 0$ is nearer to the origin.

Q-3: Find the equation of a line with slope 2 and the length of perpendicular from the origin equal to $\sqrt{5}$.

Sol:

Let c be the intercept on y-axis.
Then the equation of the lines is $y = 2x + c$

$$-2x + y = c$$

$$-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{1}{\sqrt{(2)^2+1^2}}y = \frac{c}{\sqrt{(-2)^2+1^2}}$$

(Dividing both side by $\sqrt{(-2)^2+1^2}$)

$$-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{c}{\sqrt{3}}$$ which is the normal form of (1)

Therefore RHS Denotes the length of the perpendicular from the origin. But the length of the perpendicular from the origin is $\sqrt{5}$.

Therefore, $\frac{c}{\sqrt{5}} = \sqrt{5} \Rightarrow c = 5$

Putting $c = 5$ in (1) we get $y = 2x + 5$.

which is the required eqn of the required line

Q-4: Find eqn of the line which passes through P(1, 2) and parallel to the line $x + 2y + 3 = 0$

Sol

The given line is: $x + 2y + 3 = 0$.

so slope, $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{1}{2}$

since the required line is parallel to the given line.

Equation of the required line passes through P(1, 2) and $m = -1/2$ is

$$y - y_1 = m(x - x_1)$$

Taking $m = -1/2$

$$y - 2 = \frac{-1}{2}(x - 1)$$

$$2y - 4 = -x + 1$$

$$x + 2y - 5 = 0$$

Q-5: Find equation of the line which passes through (2, 3) and perpendicular to the line $3x + 2y + 5 = 0$

Sol: The eqn of given line is $3x + 2y + 5 = 0$

slope = $m_{given} = \frac{-3}{2}$

since required line is perpendicular to the given lines.

therefore $m_{required} = \frac{-1}{m_{given}} = \frac{2}{3}$

So the equation of the required line which passes through (2, 3) and slope '2/3' is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{2}{3}(x - 2)$$

$$3y - 9 = 2x - 4$$

$$2x - 3y + 5 = 0$$

Equation of a line parallel to a given line:

Let m be the slope of the line $ax + by + c = 0$

Then slope, $m = -\frac{a}{b}$ $\left(\text{using } m = \frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$

Let c_1 be the y-intercept of the required line.

Therefore, the equation of the required line is

$$y = mx + c_1$$

$$y = -\frac{a}{b}x + c_1$$

$$ax + by - bc_1 = 0$$

$$ax + by + \lambda = 0 \quad \text{where } \lambda = -bc_1 = \text{constant}.$$

Therefore, the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$ where λ is a constant.

Equation of a line perpendicular to a given line.

Let m_1 be the slope of the given line and m_2 be the slope of a line perpendicular to the given line.

Then $m_1 = -\frac{a}{b}$ and $m_1 m_2 = -1$ (using ⊥ condition)

Therefore, $m_2 = -\frac{1}{m_1} = \frac{b}{a}$.

Let c_2 be the y- intercept of the required line. Then its equation is.

$$y = m_2 x + c_2$$

$$y = \frac{b}{a} x + c_2$$

$$bx - ay + ac_2 = 0$$

$$bx - ay + \lambda = 0 \text{ where } \lambda = ac_2 = \text{constant}.$$

Therefore the eqn of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$ where is 'λ' is a constant.

Some solved Problems :-

Q-1: Find the equation of the line which is parallel to $3x - 2y + 5 = 0$, and passes through the point $(5, -6)$.

Sol

The eqn of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x - 2y + \lambda = 0$$

The line passes through the point $(5, -6)$

Thus, $3 \times 5 - 2 \times (-6) + \lambda = 0 \Rightarrow \lambda = -27$

putting $\lambda = -27$ we get $3x + 2y - 27$.

Q2: Find the equation of the straight line that passes through the point $(3, 4)$ and perpendicular to the line $3x + 2y + 5 = 0$

Sol: The eqn of a line ⊥ to $3x + 2y + 5 = 0$ is

$$2x - 3y \lambda = 0$$

The line passes through the point $(3, 4)$

Thus $3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting $\lambda = 6$ in (1) we get $2x - 3y + 6 = 0$

Circle :-

Defination :-

A circle is the locus of a point which moves on a plane in such way that its distance from a fixed point is always constant. The fixed point is called the center of the circle and the constant distance is called radius of the circle.

1) Standard form (equation of a circle with given centre and radius)

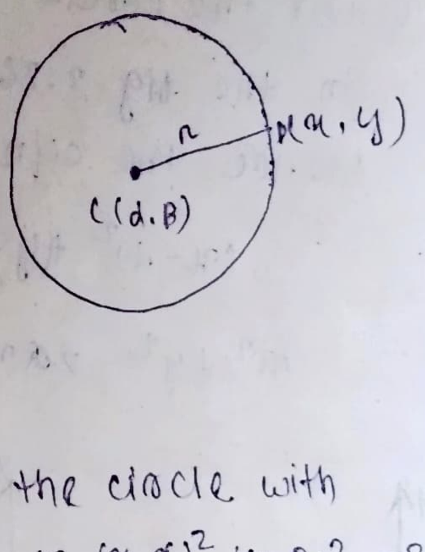
Let $C(\alpha, \beta)$ be the centre of the circle and radius of the circle be 'r'

Let $P(x, y)$ be any point on the circumference of the circle.

Then

$CP = r$

By distance formula

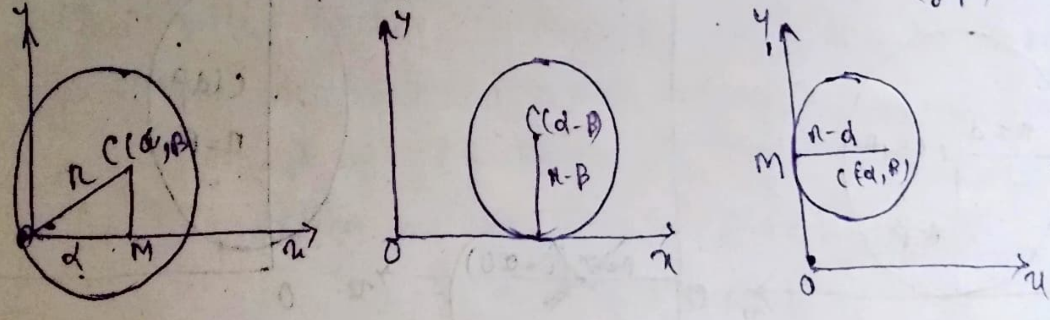


$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = r$

$(x-\alpha)^2 + (y-\beta)^2 = r^2$

Some particular cases :

The standard equation of the circle with centre at $C(\alpha, \beta)$ and radius r, is $(x-\alpha)^2 + (y-\beta)^2 = r^2$



(i) when the circle passes through the origin.

from the fig 3.24 in right angle triangle ΔOCM

$OC^2 = OM^2 + CM^2$ i.e $r^2 = \alpha^2 + \beta^2$

Then eqn (1) becomes

$$(x-\alpha)^2 + (y-\beta)^2 = \alpha^2 + \beta^2$$

$$x^2 + y^2 - 2\alpha x - 2\beta y = 0.$$

(ii) when the circle touches x-axis.

In Fig 3.25, Here, $r = \beta$

Hence, the eqn (1) of the circle becomes

$$(x-\alpha)^2 + y(\beta)^2 = \beta^2$$

$$x^2 + y^2 - 2\alpha x - 2\beta y = \alpha^2 = 0$$

(iii) when the circle touches y-axis.

In the fig 3.26 Here $r = \alpha$

Hence the eqn (1) of the circle becomes

$$(x-\alpha)^2 + (y-\beta)^2 = \alpha^2$$

$$x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$$

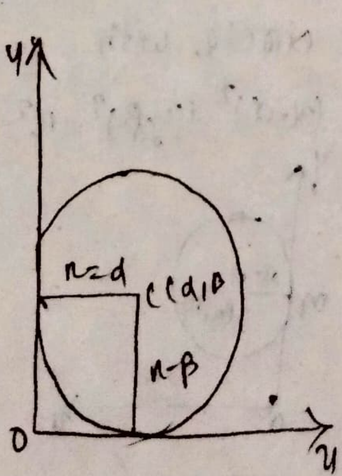


Fig 3.27

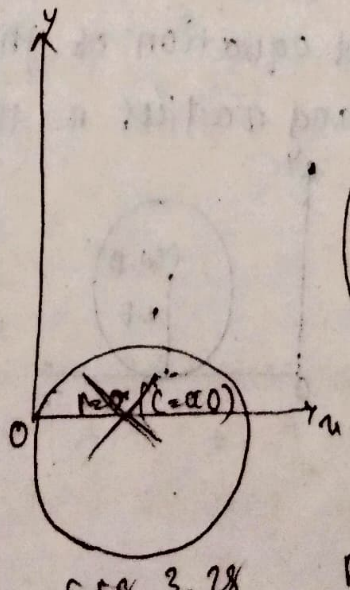


Fig 3.28

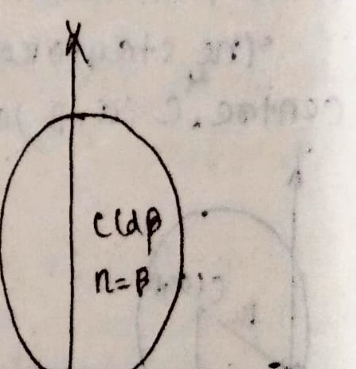


Fig 3.29

(iv) when the circle touches both the axis.

In the fig 3.27, here, $\alpha = \beta = r$

Hence the eqn(1) of the circle becomes:

$$(x-r)^2 + (y-r)^2 = r^2$$

$$x^2 + y^2 - 2rx - 2ry + r^2 = 0$$

(v) when the circle passes through the origin and centre lines on x-axis.

Here $\alpha = r$ and $\beta = 0$

Hence the eqn (1) of the circle becomes

$$(x-r)^2 + (y-0)^2 = r^2$$

$$x^2 + y^2 - 2rx = 0$$

(vi) when the circle passes through the centre lines on y-axis and origin.

Here, $\alpha = 0$ and $\beta = r$

Hence, the eqn (1) of the circle become,

$$(x-0)^2 + (y-r)^2 = r^2$$

$$x^2 + y^2 - 2ry = 0$$

Some solved problems :-

Q-1: Find eqn of the circle which has centre at (2,3) and radius is 4.

According to the standard form, the equation of circle with centre at (α, β) and radius is

$$(x-\alpha) + (y-\beta)^2 = r^2$$

∴ equation of the circle with centre at (2,3) and radius 4 is

$$(x-2)^2 + (y-3)^2 = (4)^2$$

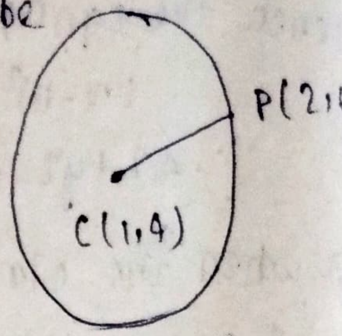
$$x^2 + y^2 - 4x - 6y + 13 = 16$$

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

Q2: Find eqn the circle which has centre at (1,4) and passes through a point (2,6)

Sol.

Given C(1,4) be the centre and r be the radius of the circle. The circle passes through the point P(2,6)



∴ PC = r

$\sqrt{(2-1)^2+(6-4)^2} = r$ (By using distance formula)

$\sqrt{1+4} = r$

$r = \sqrt{5}$

By using standard form of the circle Eqn of the circle with centre at C(1,4) and radius $\sqrt{5}$ is

$(x-1)^2+(y-4)^2 = (\sqrt{5})^2$

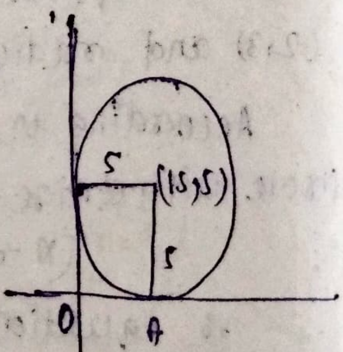
$x^2+y^2-2x-8y+1+16 = 5$

$x^2+y^2-2x-8y+12 = 0$

Q3 = find the eqn of the circle whose centre is at (5,5) and touches both the axis.

Sol :-

The centre of the given circle is at (5,5).



Since the circle touches both the axes,

∴ radius, r = 5

According to the standard form.

∴ eqn of the circle with centre at C(5,5) and radius r = 5 is

$$(x-5)^2 + (y-5)^2 = (5)^2$$

$$x^2 + y^2 - 10x - 10y + 25 = 0$$

Q = 9 If the eqn of the two diameter of a circle are $x - y = 5$ and $2x + y = 4$ and the radius of the circle is 5 find the eqn of the circle.

Sol :

let the diameters of the the circle be AB AND LM, whose equations are respectively.

$$x - y = 5 \quad \text{—— (1)}$$

and $2x + y = 4$ —— (2)

since the point of intersection of any two diameters of a circle is its centre and by solving the equations of two diameter we find the co-ordinate of the centre.

∴ solving eqn (1) and (2) we get $x = 3$ & $y = 2$

Therefore, coordinates of the centre are (3, -2) and radius is 5.

Hence equation of required circle is

$$(x-3)^2 + (y+2)^2 = 5^2$$

$$x^2 - y^2 - 6x + 4y + 9 + 4 = 25$$

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

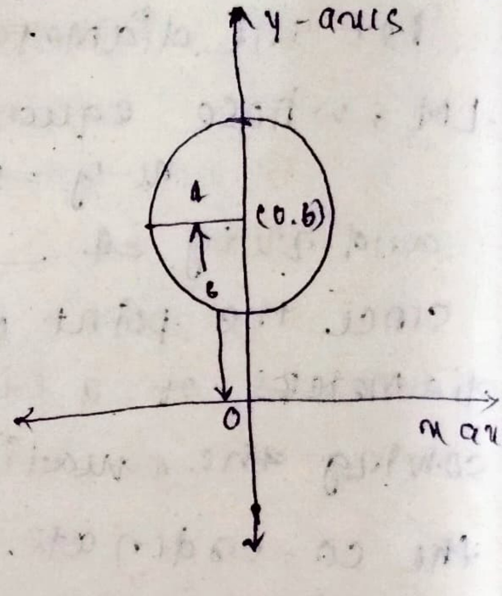
Q:5- Find the eqn of a circle whose centre lies on positive direction of y-axis at a distance 6 from the origin.

Sol:

Given, the centre of the circle lies on the y-axis at a distance 6 unit from the origin.

(∴) The centre of the circle lies at the point (C, D)

Hence, eqn of the circle with centre at (0,6) and radius '4' is



$$(x-0)^2+(y-6)^2=4^2$$

$$x^2+y^2-12y+36=16$$

$$x^2+y^2-12y+20=0$$

② General form :-

The eqn $x^2+y^2+2gx+2fy+c=0$ always represents a circle whose center is at $-(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$

Example :-

Let the equation of a circle be $2x^2+25y^2-30x-10y-6$

To find the centre and radius of the above circle, divide by coefficient of x^2 i.e. 25, as:

$$x^2+y^2-\frac{30}{25}x-\frac{10}{25}y-\frac{6}{25}=0$$

$$x^2+y^2-\frac{6}{5}x-\frac{2}{5}y-\frac{6}{25}=0$$

$$x^2+y^2-\frac{6}{5}x-\frac{2}{5}y-\frac{6}{25}=0$$

$$x^2+y^2+2\left(-\frac{3}{5}\right)+2\left(-\frac{1}{3}\right)y+\left(-\frac{6}{25}\right)=0$$

which is the general form of circle with centre at $(-g,-f)=\left(\frac{3}{5},\frac{1}{5}\right)$ and radius =

$$=\sqrt{\left(-\frac{3}{5}\right)^2+\left(-\frac{1}{5}\right)^2-\left(-\frac{6}{25}\right)}=\frac{4}{5}$$

Example-2 :-

Consider the eqn of a circle

$$x(x+y-6)=y(x-y+8)$$

or, $x^2+xy-6x=xy-y^2+8y$

$$x^2+y^2-6x-8y=0$$

which, is in the general form of circle.

$\therefore$ centre $=(-g,-f)=\left(-\frac{1}{2}\text{ coff. of } x, -\frac{1}{2}\text{ coff. of } y\right)$

$$=\left(-\frac{1}{2}(-6), -\frac{1}{2}(-8)\right)=(3,4)$$

and radius $=\sqrt{g^2+f^2-c}=\sqrt{\left(-\frac{1}{2}(-6)\right)^2+\left(-\frac{1}{2}(-8)\right)^2-0}$

$$=\sqrt{9+16-0}=5$$

Example - 3

Let the eq^n of the circle be $x^2+y^2+4x+6y+2=0$ compare this given eq^n with the general eq^n of the circle, $x^2+y^2+2gx+2fy+c=0$

Here, $2gx=4x$, $2fy=6y$ and $c=2$

Now the centre is $(-g,-f)=(-2,-3)$ and

$r=\sqrt{g^2+f^2-c}=\sqrt{-2+(-3)-c}=\sqrt{4+9-2}=\sqrt{11}$

Some Solved Problems :-

Q-1: Determine which of the circles $x^2+y^2-3x+4y$ and $x^2+y^2-6x+8y=0$ is greater.

Sol

The equations of two given circle are

$$C_1: x^2+y^2-3x+4y=0$$

and $C_2: x^2+y^2-6x+8y=0$

In 1st circle C_1, $g=\frac{3}{2}$, $f=2$, $c=0$

radius $=r_1=\sqrt{g^2+f^2-c}=\sqrt{\left(\frac{3}{2}\right)^2+2^2-0}$

$$=\sqrt{\frac{9}{4}+4}=\frac{5}{2}$$

Similarly in 2nd circle C_2, $g=3$, $f=4$, $c=0$

radius $=r_2=\sqrt{g^2+f^2-c}=\sqrt{3^2+4^2-0}=\sqrt{9+16}=5$

Since $r_1<R_2$, so the 2nd circle C_2:

$x^2+y^2-6x+8y=0$ is greater.

Q-2: Find the eqⁿ of the circle concentric with the circle $x^2+y^2-4x+6y+10=0$ and having radius 10 unit.

Sol:

The co-ordinates of the centre of the given circle

$$x^2+y^2-4x+6y+10=0 \text{ are}$$

$$\left(-\frac{1}{2}\text{ coeff. of } x \; -\frac{1}{2}\text{ coeff of } y\right)=(2,-3)$$

Since the required circle is concentric with the above circle, the centre of the required circle and given circle are same.

∴ Centre of the required circle is at $(2,-3)$

Hence the eqⁿ of the required circle with centre at $(2-3)$ and radius 10 is

$$(x-2)^2+(y+3)^2=(10)^2$$

$$x^2+y^2-4x+6y-87=0.$$

Q-3: Find the equation of the circle whose centre is at the point $(4,5)$ and passes through the centre of the circle : $x^2+y^2-6x+4y-12=0$

Sol:

The co-ordinates of the centre of the circle

$$x^2+y^2-6x+4y-12=0 \text{ are}$$

$$C\left(-\frac{1}{2}\text{ coff of } x, \; -\frac{1}{2}\text{ coeff of } y\right)=C(3,-2)$$

Therefore, the required circle passes through the point $(3,-2)$

Given, the centre of the required circle is at $(4,5)$

$\therefore$ radius of the required circle $= CC_1$

$= \sqrt{(4-3)^2+(5+2)^2} = \sqrt{1+49} = \sqrt{50}$

Hence, the eq^n of the required circle with centre at $(4,5)$ and radius $\sqrt{50}$ is

$$(x-4)^2+(y-5)^2 = (\sqrt{50})^2$$

$$x^2+y^2-8x-10y-9=0$$

Q-4: Find the equation of the circle concentric with the circle $4x^2+4y^2-24x+16y-9=0$ and having its area equal to 9π sq. units.

Sol:

The eq^n of given circle is $4x^2+4y^2-24x+16y-9=0$

$$x^2+y^2-6x+4y-\frac{9}{4}=0$$

$\therefore$ Centre $(-\frac{1}{2}\text{coeff. of } x, -\frac{1}{2}\text{coeff of } y) = (3,-2)$

Since the required circle is concentric with the above circle, the circle of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is $(3-2)$ and let its radius be 'r'.

Again, given area of the required circle $= 9\pi$

$\pi r^2 = 9\pi$

$r = 3$ units

Therefore, the eq^n of the required circle with ~~same~~ centre at $(3,-2)$ and radius '3' is

$(x-3)^2 + (y+2)^2 = (3)^2$

$x^2+y^2-6x+4y+4=0$

Q-5: Find the equation of the circle concentric with the circle $2x^2+2y^2+8x+12y-25=0$ and having the circumference equal to 6π sq. units.

Sol :-

The eqn of given circle be $= 2x^2+2y^2+8x+12y-25=0$

$x^2+y^2+4x+6y-\frac{25}{2}=0$

$\therefore$ centre $= (-2,-3)$

Since the required circle is concentric with the above circle, the centre of the required circle and above given circle are same.

$\therefore$ centere of the required circle is $(-2,-3)$

Again, Given, circumference of the required circle $= 6\pi$

$2\pi r = 6\pi$

$r = 3$ units

Therefore, the eqn of the required circle with centre at $(-2,-3)$ and radius '3' is

$(x+2)^2 + (y+3)^2 = 3^2$

$x^2+y^2+4x+6y+4=0$

Some solved problem :—

Q-1: Find eqn of the circle passes through the points (0,0), (1,0) and (0,1).

Sol:

Let the eqn of the circle be

$$x^2+y^2+2gx+2fy+C=0$$

since, the circle (1) passes through the points (0,0), (1,0) and (0,1) we have

$$0+0+0+0+C=0 \qquad C=0$$

$$1+0+2g+0+0=0 \qquad g=-1/2$$

$$0+1+0+2f+0=0 \qquad f=-1/2$$

and

putting, the values of g, f and C in eqn (1) we get

$$x^2+y^2\ 2\left(\frac{-1}{2}\right)x+2\left(\frac{-1}{2}\right)y+0=0$$

$x^2+y^2-x-y=0$, is the eqn of required circle

Q-2: Find the equation of the circle passes through the points (0,2) (3,0) and (3,2) Also find the centre and radius.

Sol:

Let the eqn of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{——} \quad (1)$$

Since the circle (1) passes through the points (0,2), (3,0), and (3,2) i.e these points lie on the circle (1), we have:

$\therefore 0+4+0+4f+c=0$ ——— (2)

or, $4f+c=-4$

$9+0+6g+0+c=0$ ——— (3)

or. $6g+c=-9$

and $9+4++6g+4f+c=0$

$6g+4f+c=-13$ ——— (4)

on solving eqn (2), (3) and (4),

Eqns (2)+(3) : $6g+4f+2c=-13$

Eqns (5)-(4) : $c=0$

putting the value of c in (2) and (3) we get,

$4f=-4$ or $f=-1$

$6g=-9$ or $g=\frac{3}{2}$

putting values of g, f and c in the general eqn of circle (1) we get,

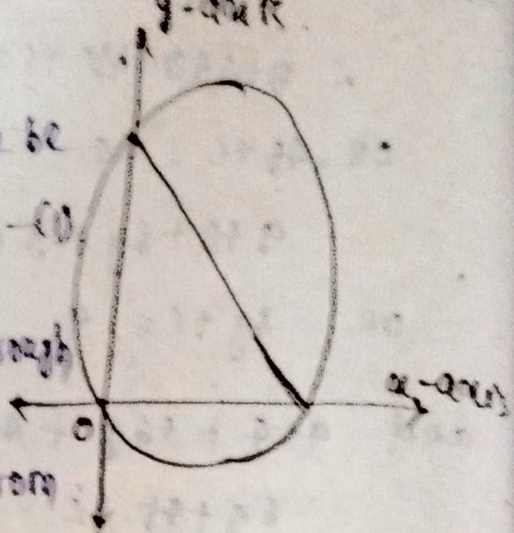
$x^2+y^2+2(-\frac{3}{2})x+2(-1)y+0=0$

$x^2+y^2-3x-2y=0$, is the equation of required circle

Now, the centre of the circle $=(-g,-f)=(\frac{3}{2},1)$

and $r=\sqrt{g^2+f^2-c}=\sqrt{\frac{9}{4}+1-0}=\frac{\sqrt{3}}{2}$

Q3: Find the equation of the ~~circ~~ circle which passes through the origin and cutts off intercept a and b from the positive parts of axes.



Sol

let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{---(1)}$$

since, the circle passes through the origin and cut off the intercept a and b from the positive axes.

So the circle passes through the points $O(0,0)$, $A(a,0)$ $B(0,b)$, we have

$$0+0+0+0+c=0, \text{ or, } c=0 \quad \text{---(2)}$$

$$a^2+0+2ag+0+0=0, \text{ or, } g=-a/2 \quad \text{---(3)}$$

and $0+b^2+0+2bf+0=0$, or, $f=-b/2$ ---(4)

putting the value of g, f and c in the eqn of circle(1), we get,

$x^2+y^2-ax-by=0$ is the req. eqn of required

Q:4: Prove that the points $(2,-4)$ $(5,-1)$ and $(0,0)$ are concyclic.

Sol:-

Note:- to prove that four given points are concyclic (i.e four points lie on the circle) we find the equation of the circle passing through any three given points and so that the 4th point lies on it.

Let the equation of the circle passing through the points (0,0), (2,-4), and (3,-1) be $x^2+y^2+2gx+2fy+c=0$

since the point (0,0) lies on circle (1) we have,

$$0+0+0+0+c=0 \quad \text{or} \quad c=0$$

Again since the point (2,-4) lies on circle (1) we have,

$$4+16+4g-8f+0=0$$

$$\text{or } 4g-8f=-20$$

$$g-2f=-5$$

Also, since the point (3,-1) lies on circle (1) we have,

$$9+1+6g-2f+0=0$$

$$\text{or, } 6g-2f=-10$$

$$\text{or, } 3g-f=-5$$

Now, solving equations (3) & (4) we get,

$$g=-1 \ \& \ f=2$$

Putting the values of g, f & c in the equation of circle (1) we get

$$x^2+y^2-2x+4y=0$$

Is the equation of circle. Now to check the

4th point (3, -3) lies on the circle (5) we put $x = 3$ and $y = -3$ in eqn (5)

$$9+9-6-12=0$$

Therefore, the point (3, -3) ~~that~~ satisfie the eqn of circle (5) and lies on the circle.

Hence, the given point are concyclic.

Q(5) Find the equation of the circle circumscribing the triangle $\triangle ABC$ whose vertices are A(1, -5), B(5, 7) C(-5, 1).

Sol :

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Since, the circle circumscribing the triangle $\triangle ABC$ with vertices A(1, -5) B (5, 7) & C(-5, 1)

So, the circle (1) passes through the point A(1, -5) B (5.7) C (-5, 1)

Therefore,

$$1+25+2g-10f+c=0$$

or, $2g-10f+c=26$

$$25+49+10g+10F+c=0$$

or, $10g+14F+c=-74$

and $25+1-10g+2f+c=-74$,

$10g+2f+c=-26$

on solving eqn (2), (3) & (4) we get,

eqn(3) - eqn(2): $8g+24f=-48$, or, $g+3f=-6$

eqn(3) - eqn (4): $20g+12f=-48$, or, $5g+3f=-12$

eqn(5) - eqn(6) - $4g=6$ or $g=-\frac{3}{2}$

i.e $-\frac{3}{2}+3f=-6$ or $f=-\frac{3}{2}$

i.e $2(-\frac{3}{2})-10(\frac{3}{2})+c=-26$ or $c=-38$

putting the values of g, f and c in the eqn of circle (1)

$$x^2+y^2+2(-\frac{3}{2})x+2(-\frac{3}{2})y+(-38)=0$$

or $x^2+y^2-3x-3y-38=0$ is the eqn of required circle.

All the physical quantities can be divided into two types.

(i) Sclar quantity or scalar.
(ii) Vector quantity or vector

<u>Scalar Quantity</u> :-

The physical quantity which requires only magnitude for its complete specification is called as scalar quantites.

<u>Examples</u> : speed, Mass, distance, volume. etc.

<u>VECTOR</u> :

A directed line segment is called as vector.

<u>VECTOR QUANTITY</u> :

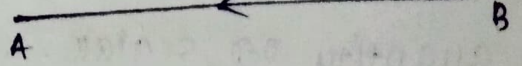
A physical quantity which requires both magnitude & direction for its complete specification and satisfies the law of vector addition is called as vector quantities.

<u>Ex</u> : Displacement, force, acceleration, velocity, momentum etc.

<u>REPRESENTATION OF VECTOR</u> :-

A vector is a directed line segment $\vec{AB}$ where A is the initial point and B is the terminal point and direction is from A to B.

similarly $\overrightarrow{BA}$ is a directed line which represents a vector having intial point B and terminal point A.



Magnitude of a vector : Magnitude or modulus of a vector is the length of the vector. It is a scalar quantity.

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}|$ = Length $AB = AB$.

TYPES OF VECTOR :

(1) Null vector OR zero vector or void vector :

A vector having zero magnitude and arbitary direction is called as a null vector and is denoted by $\bar{0}$.

clearly, a null vector has no definite direction. If $\vec{a} = \overrightarrow{AB}$, then $\vec{a}$ is a null (or zero) vector iff $|\vec{a}| = 0$. i.e if $|\overrightarrow{AB}| = 0$.

For a null vector initiad and terminal points are same.

(2) PROPER VECTOR :-

Any non zero vector is called as a proper vector. If $|\vec{a}| \neq 0$ then $\vec{a}$ is a proper vector.

(3) unit vector :

A vector whose magnitude is unity is called a unit vector. unit vectors are denoted by a small letter with ^ over it.

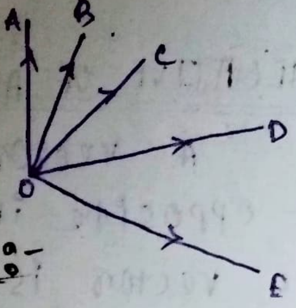
for example $a \cdot |a| = 1$

NOTE : The unit vector along the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

(4) CO-INITIAL VECTORS :

Vectors having the same intial points are called co-initial vector.

In, $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$ and $\vec{OE}$, are co-initial vector.

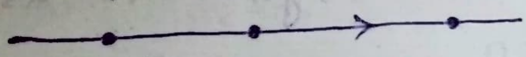


(5) LIKE AND UNLIKE VECTORS :-

VECTORS are said to be like if they have same direction, and unlike if they have opposite direction.

(6) CO-LINEAR VECTORS :-

Vectors are said to be co-linear or parallal if they have the same line of action. $\vec{AB}$ and $\vec{BC}$ are co-linear.



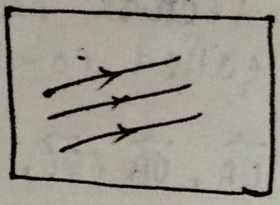
(7) PARALLEL VECTORS :-

Vectors are said to be parallel if they have same line of action or have line of action parallel to one another.

In the vectors are parallel to each other.

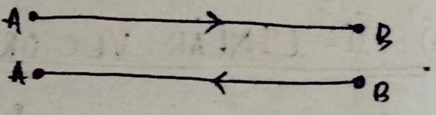
(8) CO-PLANER VECTORS :-

Vectors are said to be co-planer if they lies on the same plane. In vector $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planer.



(9) NEGATIVE OF A VECTOR :

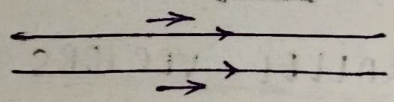
A vector having same magnitude but opposite in direction to that of a given vector is called negative of a vector. If $\vec{a}$ is any vector than negative vector of it is written as $-\vec{a}$ and $|\vec{a}| = |-\vec{a}|$ but both have direction opposite to each other as show in.



(10) EQUAL VECTORS :-

Two vectors are said to be equal if they have same magnitude as well as same direction.

Thus $\vec{a} = \vec{b}$



Remarks : Two vectors can not be equal

(i) If they have different magnitude

(ii) If they have inclined supports.

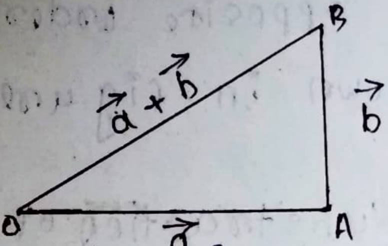
(iii) If they have different sense.

VECTOR OPERATIONS :-

ADDITION OF VECTORS :-

Tringle law of vector addition :-

The law state that if two vectors are represented by the two side of a tringle taken in same order their sum or resultant is represented by the 3rd side of the tringle with the direction in reverse order.



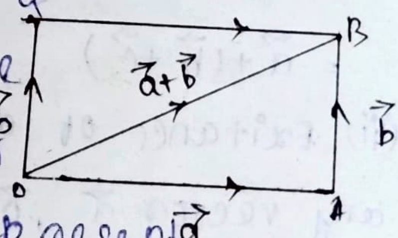
Note-1 = The Method of drawing a tringle in order to define the vector sum $(\vec{a}+\vec{b})$ is called tringle law of addition of the vectors.

Note-2 :-

Since any side of a tringle is less then the sum of the other two sides.

Parallelogrom law of vector addition :-

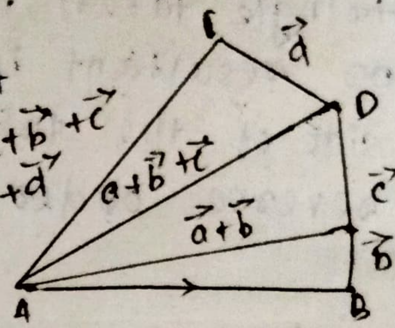
If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogrom in magnitude and direction then their sum (resultant) is representated in magnitude and direction by the



digonal which is passing through the common initial point of the two vectors.

Polygon law of vector addition :-

If $\vec{a}$, $\vec{b}$, and $\vec{d}$ are the four sides of a polygon in same order. order then their sum is represented by the last side of the polygon take $\vec{a}+\vec{b}+\vec{c}+\vec{d}$ in opposite order as shown in figure.



Substraction of two vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the substraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a}-\vec{b}$ is defined as the addition of $-\vec{b}$ with $\vec{a}$ i.e $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Properties of vector addition :-

(i) vector addition is commutative i.e. If $\vec{a}$ & $\vec{b}$ are any two vector then :- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) vector addition is asscociative i.e If $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors then $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) Exitance of additive identity i.e. for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e $\vec{0}+\vec{a}=\vec{a}+\vec{0}=\vec{a}$, where $\vec{0}$ is a null vector.

(v) Existance of additive inverse :- If $\vec{a}$ is any non zero vector then $-\vec{a}$ is the additive inverse of $\vec{a}$, so that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Multiplication of a vector by a scalar :-

If $\vec{a}$ is a vector and K is a non zero scalar then the multiplication of the vector $\vec{a}$ by the scalar K is a vector denoted by $K\vec{a}$ or $\vec{a}K$ whose magnitude $|K|$ times that of $\vec{a}$.

i.e $K\vec{a} = |K| \times |\vec{a}|$

$= K \times |\vec{a}|$ if $K \geq 0$

$= (-K) \times |\vec{a}|$ if $K < 0$

The direction of $K\vec{a}$ is same as that of $\vec{a}$ if K is positive; and opposite as that of $\vec{a}$ if K is negative.

$K\vec{a}$ and $\vec{a}$, are always parallel to each other.

Properties of scalar Multiplication of vectors:-

If h and K are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

(i) $K(\vec{a} + \vec{b}) = K\vec{a} + K\vec{b}$

(ii) $(h+K)\vec{a} = h\vec{a} + K\vec{a}$ (Distributive law)

(iii) $(h+K)\vec{a} = h(K\vec{a})$ (Associative law)

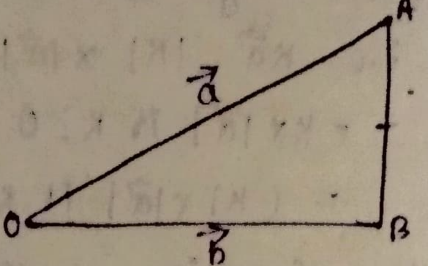
(iv) $1 . \vec{a} = \vec{a}$

(v) $0 . \vec{a} = \vec{0}$

Position vector of a point :-

Let O be a fixed point called origin, let P be any other point, then the vector $\vec{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{P}$.

As shown in figure-13, let AB be any vector, then applying triangle law of addition we have



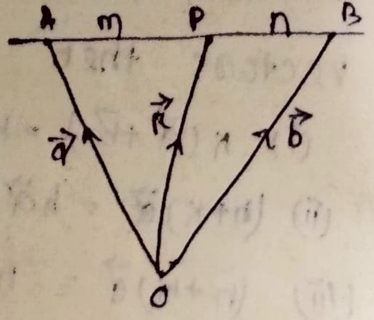
$\vec{OA} + \vec{AB} = \vec{OB}$ where $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

$\Rightarrow \vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

= (position vector of B) - (position vector of A)

Section formula :-

Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB, dividing it in the ration m:n internally Then the position vector of P i.e $\vec{r}$ is given by the formula $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$



If P is diveded AB externally in the ration m:n then $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

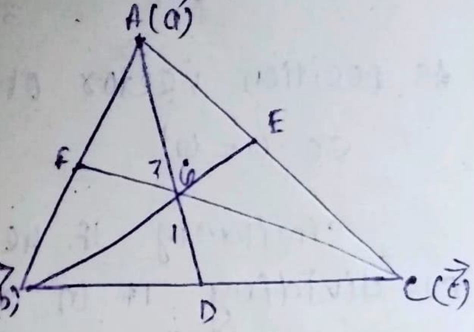
If P is the midpoint of AB then $\vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$

Example-1

prove that by vector method the medians of a tringle are con current.

Solution :-

Let ABC be a tringle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A, B and C respectively. we have to show that the medians of this tringle are concurrent



Let AD, BE and CF are the three medians of the tringle.

Now as D be the midpoint of BC, so position vector of D i.e $\vec{d} = \dfrac{\vec{b}+\vec{c}}{2}$

Let G be any point of the median AD which divides AD in the Ratio 2:1 Then position vector of G is given by

$$\vec{g} = \frac{2\vec{d}+\vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1\vec{a}}{3}$$ (by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Let G' be a point which divides BE in the ratio 2:1

position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$

Then position vector of G' is given by

$$\vec{g'} = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{2\left(\frac{\vec{a}+\vec{c}}{2}\right)+1\vec{b}}{3}$$ (by applying section formula)

$$\Rightarrow \vec{g'} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

As position vector of a point is unique

so G = G'

Similarly if we take G'' be a point on CF dividing it in 2:1 ratio then the position vector of G'' will be same as that of G.

Hence G is the one point where three Median meet.

∴ The three Medians of a triangle are concurrent (proved)

<u>example - 2 :-</u>

Prove that (i) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

(ii) $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$

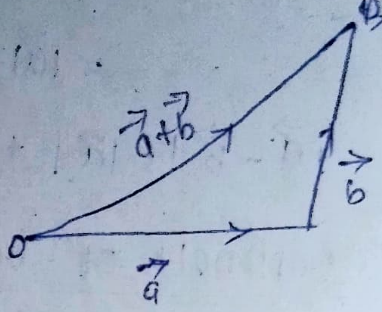
(iii) $|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Proof :- Let O, A and B be three points

which are not collinear and then draw a tringle OAB.

let $\vec{OA} = \vec{a}$; $\vec{AB} = \vec{b}$

then by tringle law of addition we have $\vec{OB} = \vec{a} + \vec{b}$



From properties of tringle we know that the sum of any two sides of a tringle is greater than the third side.

$\Rightarrow OB < OA + AB$

$\Rightarrow |\vec{OB}| < |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$ ——— (1)

when O, A, B are collinear then form figure -17 it is clear that

$OB = OA + AB$

$\Rightarrow |\vec{OB}| = |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ ——— 2

From (1) and (2) we have

$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

(ii) $|\vec{a}| = |\vec{a} - \vec{b} + \vec{b}|$ ——— (1)

But $|(\vec{a} - \vec{b}) + \vec{b}| \leq |\vec{a} - \vec{b}| + |\vec{b}|$ (from tringle inequality) ——— (2)

from (1) and (2) we get $|\vec{a}| \leq |\vec{a} - \vec{b}| + |\vec{b}|$

$\Rightarrow |\vec{a}| - |\vec{b}| \leq |\vec{a} - \vec{b}|$ (proved)

(iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \leq |\vec{a}| + |-\vec{b}|$ (from triangle inequality)

$= |\vec{a}| + |\vec{b}|$ (as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| = |\vec{a}| + |\vec{b}|$ (proved)

Componels of vecter in 2D

Let xoy be the co-ordinate plane and P(x,y) be any point in this plane.

The unit vector along direction of x axis i.e $\vec{OX}$ is denoted by $\hat{i}$.

The unit vector along direction of y axis i.e $\vec{OY}$ is denoted by $\hat{j}$.

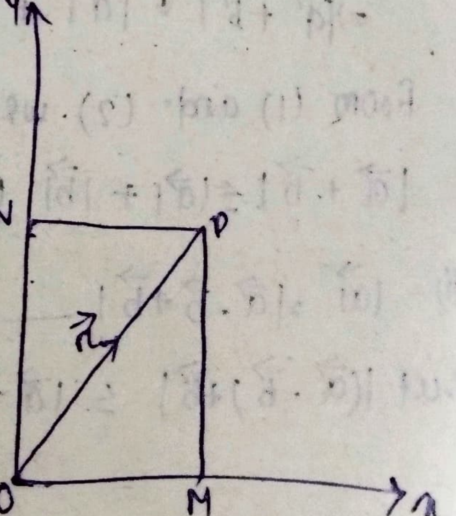
Then from figure-18 It is clear that

$\vec{OM} = x\hat{i}$ and

$\vec{ON} = y\hat{j}$

So, the position-vector of P is given by

$$\boxed{\vec{OP} = \vec{r} = x\hat{i} + y\hat{j}}$$



And $OP = |\vec{OP}|$

$= r = \sqrt{x^2+y^2}$

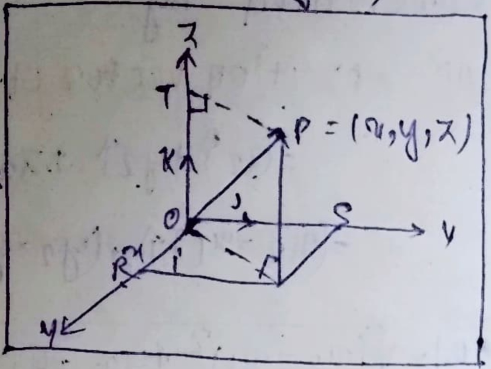
Representation of vector in component form in 2D :-

If $\vec{AB}$ is any vector having end points A (x_1, y_1) and $B(x_2, y_2)$, then it can be represented by $\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$.

Components of vector in 3D :-

Let $P(x, y, z)$ be a point in space and $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along x axis, y axis and z axis respectively (as shown in fig-19)

The the position vector of p is given by $\vec{OP} = x\hat{i} + y\hat{j} + \hat{k}$ the vectors $x\hat{i}, y\hat{i}, z\hat{k}$ are called the components of $\vec{OP}$ along x-axis, y-axis and z-axis respectively.



And $OP = |\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$

Addition and scalar multiplication in terms of component form of vectors.

form any vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}\,(a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{j} + (a_2 - b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(iii) $k\vec{a} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$ where k is a scalar

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$

REPRESENTATION OF VECTOR IN COMPONENT FORM IN 3-D & DISTANCE BETWEEN TWO POINTS :-

If $\vec{AB}$ is any vector having end points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then it can be represented by

$\vec{AB}$ = position vector of B − position vector of A

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} - y_1\hat{j} + z_1\hat{k})$

$= (x_2 - x_1)\hat{i}, (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example -3

Show that the points A(2, 6, 3), B (1, 2, 7) and C (3, 10, −1) are collinear.

Solution :-

From given data position vector of A, $\vec{OA} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

position vector of B, $\vec{OB} = \hat{i} + 2\hat{j} + 7\hat{k}$

position vector of C, $\vec{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$

Now $\vec{AB} = \vec{OB} - \vec{OA} = (1-2)\hat{i} + (2-6)\hat{j} + (7-3)\hat{k}$

$= \hat{i} - 4\hat{j} + 4\hat{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k}$

$= \hat{i} + 4\hat{j} - 4\hat{k} = -(-\hat{i} - 4\hat{j} + 4\hat{k}) = -\vec{AB}$

$\Rightarrow \vec{AB} \parallel \vec{AC}$ or collinear.

$\therefore$ They have same support and common point A.

As 'A' is common to both vector, that proves A, B and C are collinear.

Example - 4

Prove that the points having position vector given by $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form a right angled triangle.

Solution:-

Let A, B and C be the vertices of triangle with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively

Then, $\vec{AB}$ = position vector of B − position vector of A

$= (1-2)\hat{i} + (-3-(-1))\hat{j} + (-5-1)\hat{k} = \hat{i} - \hat{j} - 6\hat{k}$.

$\vec{BC}$ = position vector C − position vector B.

$= (3-1)\hat{i} + (-4-(-3))\hat{j} + (-4-(-5))\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{AC}$ = position vector of C − position vector of A.

$= (3-2)\hat{i} + (-4(-1))\hat{j} + (-4-1)\hat{k} = \hat{i} - 3\hat{j} - 5\hat{k}$

Now $AB = |\vec{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

$BC = |\vec{BC}| = \sqrt{(2)^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$AC = |\vec{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

from above $BC^2 + AC^2 = 6 + 35 = 41 = AB^2$

Hence ABC is right angled triangle

c) Example - 5 :-

Find the unit vector in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j} + \hat{k}$.

Ans :- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} - 4\hat{j} + \hat{k}}{\sqrt{3^2 + (-4)^2 + 1^2}} = \frac{3\hat{i} - 4\hat{j} + \hat{k}}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}\hat{i} - \frac{4}{\sqrt{26}}\hat{j} + \frac{1}{\sqrt{26}}\hat{k}.$$

Example - 6 :-

Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$.

Ans :-

Let $\vec{r} = \vec{a} + \vec{b} = (\hat{i} + \hat{j} - \hat{k}) + (\hat{i} - \hat{j} + 3\hat{k}) = 2\hat{i} + 2\hat{k}$.

unit vector along direction of $\vec{a} + \vec{b}$ give by

$$= \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{i} + 2\hat{k}}{\sqrt{2^2 + 2^2}} = \frac{2\hat{i} + 2\hat{k}}{\sqrt{8}}$$

$$= \frac{2}{\sqrt{8}}\hat{i} + \frac{2}{\sqrt{8}}\hat{k}$$

$$= \frac{2}{2\sqrt{2}}\hat{i} + \frac{2}{2\sqrt{2}}\hat{j} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

Angle between the vectors :-

Dot product or scalar product of vectors :-

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = ab\cos\theta$$

Angle betn two non-zero vectors :-

$$\cos\theta = \frac{\vec{a}\vec{b}}{ab} = \vec{a} \cdot \vec{b} \cdot \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$\theta = \cos^{-1}\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Condition of perpendicularity :-

$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

Condition of parallelism

Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are to each other $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Example : 7 : Find the value of p for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$, $\hat{i} + p\hat{j} + 3\hat{k}$ are perpendicular to each other.

Solution :-

Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$

$\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

Here $a_1 = 3, a_2 = 2, a_3 = 9$

Examples - 27

Find the value of P for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$, $\hat{i} + p\hat{j} + 3\hat{k}$ are perpendicular to each other.

Solution :- Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ & $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

Here $a_1 = 3, a_2 = 2, a_3 = 9$

$b_1 = 1, b_2 = P, b_3 = 3$

Given $\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

$= 3.1 + 2.P + 9.3 = 0$

$= 3 + 2P + 27 = 0$

$= 2P = -30 \Rightarrow$ $P = -15$ (ANS)

(Q: 8) Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel to each other (2014 - W)

Solution :-

Given $\vec{a} \parallel \vec{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$

(taking 1st two terms)

$\Rightarrow 3 = \frac{2}{p} = p = \frac{2}{3}$ (Ans)

{ Note : any two expression may be taken for finding p }.

Q.9) Find the scalar product of $3\hat{i} - 4\hat{j}$ and $2\hat{j} + \hat{i}$.

Solution

$(3\hat{i} + 4\hat{j}) \cdot (-2\hat{i} + \hat{j}) = (3 \times (-2) + ((-4) \times 1))$

$\Rightarrow -6 + (-4) = -10$

(Q.10) Find the angle betn the vectors $5\hat{i} + 3\hat{j} + 4\hat{k}$ and $6\hat{i} - 8\hat{j} - \hat{k}$.

Solution : let $\vec{a} = 5\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 6\hat{i} - 8\hat{j} - \hat{k}$

let θ be the angle betn $\vec{a}$ and $\vec{b}$

Then $\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

$\Rightarrow \cos^{-1} = \frac{5 \cdot 6 + 3 \cdot (-8) + 4(-1)}{\sqrt{5^2 + 3^2 + 4^2}\sqrt{6^2 + (-8)^2 + (-1)^2}}$

$\Rightarrow \cos^{-1}\left(\frac{30-24-4}{\sqrt{50}\sqrt{101}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{50}\sqrt{101}}\right)$

Q=11 Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where,

$\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$

Solution :- scalar projection of $\vec{a}$ on $\vec{b}$ =

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{1.3+(-1).1+(-1).3}{\sqrt{(3^2+1^2+3^2)^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$$

vector projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{b}$ =

$$\frac{1.3+(-1).1+(-1).3}{\sqrt{(3^2+1^2+3^2)^2}} = (3\hat{i}+\hat{j}+3\hat{k})$$

$$\Rightarrow \frac{3-1-3}{19}(3\hat{i}+\hat{j}+3\hat{k}) = \frac{-1}{19}(3\hat{i}+\hat{j}+3\hat{k})$$

Q:12

Find the scalar and vector projection of $\vec{b}$ on $\vec{a}$ where, $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}-4\hat{k}$

Solution scalar projection of $\vec{b}$ on $\vec{a}$:

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3.2+1.3+(-2)(-4)}{\sqrt{3^2+1^2+(-2)^2}} = \frac{6+3+8}{\sqrt{14}} = \frac{17}{\sqrt{14}}$$

vector projection of $\vec{b}$ on $\vec{a}$ =

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{3.2+1.3+(-2).-(-4)}{\sqrt{3^2+1^2+(-2)^2}} = (3\hat{i}+\hat{j}-2\hat{k})$$

$$\Rightarrow \frac{17}{14}(3\hat{i}+\hat{j}-2\hat{k})$$

Q 13 :- If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or, $\vec{a} \perp (\vec{b} - \vec{c})$.

proof = Given $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b} - \vec{c}) = \vec{0}$

(applying distributive property)

Dot product of above two vector is zero indicates the following conditions.

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ (proved)

Example - 14:-

Find the work done by the force $\vec{F} = \hat{i} + \hat{j} - \hat{k}$ acting on a particle if the particle is displace from A(3, 3, 3) to B (4, 4, 4)

Ans let O be the origin, then position vector of A $\vec{OA} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

position vector of B $\vec{OB} = 4\hat{i} + 4\hat{j} + 4\hat{k}$

Then displacement is given by

$\vec{d} = \vec{AB} = |\vec{OB} - \vec{OA}| = (4\hat{i} + 4\hat{j} + 4\hat{k}) - (3\hat{i} + 3\hat{j} + 3\hat{k})$

$= (\hat{i} + \hat{j} + \hat{k})$

so work done by the force W =

$\vec{F} . \vec{d} = \vec{F} . \vec{AB} =$

$(\hat{i} + \hat{j} - \hat{k}) . (\hat{i} + \hat{j} + \hat{k})$

$= 1.1 + 1.1 + (-1) . 1 = 1$ units.

Example-15 If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them then prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}.\vec{b}|$

Proof $(|\vec{a}-\vec{b}|)^2 = (\vec{a}-\hat{b}).(\vec{a}-\hat{b}) = (\hat{a}.\hat{a}) - (\vec{a}.\hat{b}) + (\hat{b}.\hat{b})$

(Distributive property)

$\Rightarrow (|\hat{a}|^2 - (\vec{a}.\hat{b}) - (\vec{a}.\hat{b}) + |\hat{b}|^2$ commutative property

$\Rightarrow 1^2 - 2\vec{a}.\hat{b} + 1^2$ { as $\vec{a}$ and $\hat{b}$ are unit vectors so their magnitude are 1 }

$\Rightarrow 2 - 2\vec{a}.\hat{b} = 2(1 - \vec{a}\hat{b})$

$\Rightarrow 2(1 - |\vec{a}|.|\vec{b}|\cos\theta)$

$\Rightarrow 2(1 - 1.1\cos\theta)$

$\Rightarrow 2(1 - \cos\theta) = 2.2\sin^2\frac{\theta}{2}$

Taking square root of both sides we have

$|\vec{a} - \hat{b}| = 2\sin\frac{\theta}{2}$

$\Rightarrow \sin\frac{\theta}{\sqrt{2}} = \frac{1}{2}|\vec{a}.\hat{b}|$ (proved)

Example-16

If the sum of two unit vectors, then show that the magnitude of their difference is $\sqrt{3}$

proof = $\vec{a}, \hat{b}$ and $\hat{c}$ are three units vectors such that $\hat{a} + \hat{b} = \hat{c}$

squaring both side we have,

$\Rightarrow (|\hat{a}| + |\hat{b}|)^2 = (|\hat{c}|)^2$

$\Rightarrow (|\hat{a}|)^2 + (|\vec{b}|)^2 + 2.\hat{a}.\hat{b} = 12$

$\Rightarrow 1^2 + 1^2 + 2|\hat{a}||\hat{b}|\cos\theta = 1$ {where θ is the angle betn $\hat{a}$ and $\hat{b}$}

$\Rightarrow 1 + 1 + 2\cos\theta = 1$

$\Rightarrow 2\cos\theta = -1$

$\Rightarrow \cos\theta = \frac{-1}{2}$

Now we have to find the magnitude of their difference i.e $\hat{a} - \hat{b}$

so, $(|\vec{a} - \vec{b}|)^2 = (|a|)^2 + (|b|)^2 - 2\hat{a}\hat{b} = 1^2 + 1^2 - 2|\hat{a}||\hat{b}|\cos\theta$

$\Rightarrow 2 - 2\cos\theta = 2 - 2\left(-\frac{1}{2}\right) = 2 - (-1) = 3$

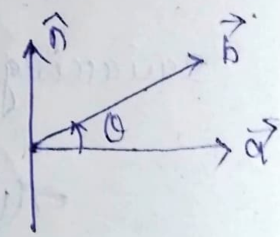
$(\therefore) |\hat{a} - \hat{b}| = \sqrt{3}$ (proved)

Vector product or cross product :-

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then the vector product of these two vectors denoted by $\vec{a} \times \vec{b}$ is defined as

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \sin\theta \, \hat{n}}$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.



the direction of $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$.

Properties of cross product:-

(i) vector product is not commutative $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.

(ii) for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(iii) for any scalar m, $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

(iv) Distributive $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(v) Vector product of two parallel or collinear vector is zero.

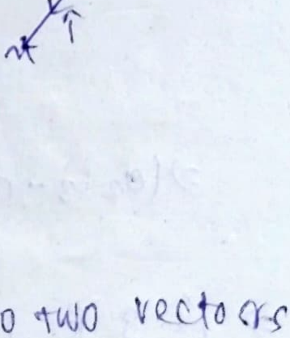
$\vec{a} \times \vec{a} = \vec{0}$ and if $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$ {as $\theta = 0$ or $180° \Rightarrow \sin\theta = 0$}

using this property we have,

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

(v) vector product of orthonormal units vectors from a right handed system.

the three mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$ form a right handed system i.e. $\hat{i} \times \hat{j} = \hat{k} = -(\hat{j} \times \hat{i})$



(as $\theta = 90°$, then $\sin\theta = 1$)

$\hat{j} \times \hat{k} = \hat{i} = -(\hat{k} \times \hat{j})$

$\hat{k} \times \hat{i} = \hat{j} = -(\hat{i} \times \hat{k})$

unit vector perpendicular to two vectors:-

unit vector perpendicular to two gives vectors $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Angle between two vectors :-

let θ be the angle betn $\vec{a}$ and $\vec{b}$, then $\vec{a} \times \vec{b} = (|\vec{a}| \cdot |\vec{b}| \sin\theta)\,\hat{n}$

Taking modulo of both sides we have

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta$$

$$\Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \cdot |\vec{b}|}$$

Hence $= \sin^{-1}\left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \cdot |\vec{b}|}\right\}$

Vector product in component form :-

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} + \hat{j})\, a_1b_3(\hat{i} + \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) +$
$a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k})$

$= a_3b_1 = (\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k})$

using properties $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{i} \times \hat{j} = \hat{k} = -(\hat{j} \times \hat{i})$

$\hat{j} \times \hat{k}) = \hat{i} = -(\hat{k} \times \hat{j})$ and

$\hat{k} \times \hat{i} = \hat{j} = -(\hat{i} \times \hat{k})\}$

$\Rightarrow (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{i.e. } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & a_3 \end{vmatrix}$$

Example - 17 :-

If $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{k}$ then find $|\vec{a} \times \vec{b}|$

Ans: we have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \{(3\times3) - (0\times(-2))\}\hat{i} - \{(1\times3) - (-1)\times(-2)\}\hat{j} + \{(1\times0) - (1\times3)\}\hat{k}$

$= 9\hat{i} - \hat{j} + 3\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81 + 1 + 9} = \sqrt{91}$

Ex : 18 - Determine the area of the parallelogram whose adjcent sides are the vectors $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{j}$:

Ans = Area of the parallelogram with adjcent sides given by $\vec{a}$ and $\vec{b}$ given by:

area $= |\vec{a} \times \vec{b}| = |2\hat{i} \times 3\hat{j}| = |6\hat{k}| = 6$ sq units (Ans)

Ex - 19 find a unit vector perpendiculer to both the vectors $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 3\hat{k}$.

Ans: unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ ——— (1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix}$

$= (3-1)\hat{i} - (6+3)\hat{j} + (-2-3)\hat{k}$

$= 2\hat{i} - 9\hat{j} - 5\hat{k}$ ——— (2)

From (1) and (2) we have,

$\hat{n} = \dfrac{2\hat{i} - 9\hat{j} - 5\hat{k}}{|2\hat{i} - 9\hat{j} - 5\hat{k}|} = \dfrac{2\hat{i} - 9\hat{j} - 5\hat{k}}{\sqrt{2^2 + (-9)^2 + (-5)^2}} = \dfrac{2\hat{i} - 9\hat{j} - 5\hat{k}}{\sqrt{110}}$

$\Rightarrow \dfrac{2}{\sqrt{110}}\hat{i} - \dfrac{9}{\sqrt{110}}\hat{j} - \dfrac{5}{\sqrt{110}}\hat{k}$ (Ans)

Example - 20 :-

If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$, then find the sine of the angle between these vectors.

Ans :- We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \cdot |\vec{b}|}$ ——— (1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$

$= -3\hat{i} + 5\hat{j} + 11\hat{k}$

Hence $|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9+25+121} = \sqrt{55}$ ——— (2)

Again $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$ ——— (3)

and $|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9+16+1} = \sqrt{26}$ ——— (4)

from equation (1), (2), (3) & (4) we have

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \cdot |\vec{b}|} = \frac{\sqrt{55}}{\sqrt{6}\cdot\sqrt{26}} = \frac{\sqrt{56}}{\sqrt{156}}$ (Ans)

Q - 21 Calculate the area of the triangle ABC (by vector method) where A(1, 1, 2) B(2, 2, 3) and C(3, -1, -1)

Soln :- Let the position vector of the vertices A, B and C is given by $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively,

Then $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{c} = 3\hat{i} + -\hat{j} - \hat{k}$

Now $\vec{AB}$ = position vector of B - position vector A

$= 2\hat{i} + 2\hat{j} + 3\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$= (3-1)\hat{i} + (2-1) + (3-2)\hat{k}$

$= \hat{i} + \hat{j} + \hat{k}$

Similarly $\vec{AC}$ = position vector C − position vector A

$= 3\hat{i} - \hat{j} - \hat{k} - (\hat{i} + \hat{j} + \hat{k})$

$= (3-1) + (-1-1) + (-1-2)\hat{k}$

$= 2\hat{i} - 2\hat{j} - 2\hat{k}$

Now $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= (-3+2)\hat{i} - (-3-2)\hat{j} + (-2-2)\hat{k} = \hat{i} + 5\hat{j} - 4\hat{k}$

Hence area of the triangle is given by

$\Delta = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{(-1)^2 + 5^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{1+25+16} = \frac{1}{2}\sqrt{42}$ sq units.

Example - 22

Find the area of a parallelogram whose diagonals are determined by the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

Ans:- Area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ are given by $A = \frac{1}{2}|\vec{a} \times \vec{b}|$

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k}$

$= 2\hat{i} - 14\hat{j} - 10\hat{k}$

Now $A = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$

$= \frac{1}{2}\sqrt{4+196+100}$

$= \frac{\sqrt{300}}{2}$

$= \frac{10\sqrt{3}}{2}$

$= 5\sqrt{3}$ Sq unit (Ans)

Ex-23 for any vector $\vec{a}$ and $\vec{b}$, prove that $(\vec{a}\times\vec{b})^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2$ where a and b are magnitude of $\vec{a}$ and $\vec{b}$ respectively.

proof:- $(\vec{a}\times\vec{b})^2 = (|\vec{a}|\cdot|\vec{b}|\sin\theta\hat{n})^2$

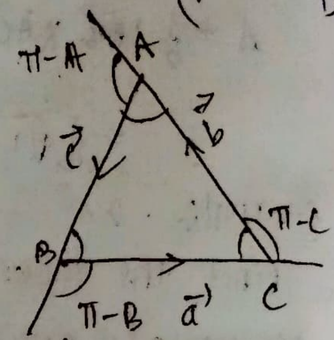
$$= (ab\sin\theta\hat{n})^2 = a^2b^2\sin^2\theta$$

$$= a^2b^2(1-\cos^2\theta) = a^2b^2 - a^2b^2\cos\theta$$

$$= a^2b^2 - (ab\cos\theta)^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2 \text{ (proved)}$$

Example-24

In ΔABC, prove by vector method that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
where $BC = a$, $CA = b$ and $AB = c$



proof :- As shown in figure-25 ABC is a triangle having $\vec{a} = \vec{BC}$, $\vec{b} = \vec{CA}$ and $\vec{c} = \vec{AB}$
from triangle law of vector we know that

$$\vec{BC} + \vec{CA} = \vec{BA}$$

$$\Rightarrow \vec{a}+\vec{b} = -\vec{c}$$

$$\Rightarrow \vec{a}+\vec{b}+\vec{c} = \vec{0} \text{ ——— (1)}$$

(taking cross product of both side with $\vec{a}$ we get)

$$\Rightarrow \vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times\vec{0}$$

$$\Rightarrow (\vec{a}\times\vec{a}) + (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) = \vec{0}$$

$$\Rightarrow \vec{0} + (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) = 0$$

$$\Rightarrow (\vec{a}\times\vec{b}) = -(\vec{a}\times\vec{c})$$

$$\Rightarrow (\vec{a}\times\vec{b}) = (\vec{c}\times\vec{a}) \text{ ——— (2)}$$

similarly taking cross product with $\vec{b}$ both sides (1) we have.

$= (\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c})$ ——— (3)

from (2) & (3), $(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a})$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

$\Rightarrow ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

As from fig-25 it is clear that angle betn $\vec{a}$ & $\vec{b}$ is $\pi - C$, $\vec{b}$ and $\vec{c}$ is $\pi - A$ and $\vec{c}$ and $\vec{a}$ is $\pi - B$.

Dividing above equation by abc we have

$$\Rightarrow \frac{ab\sin(\pi - C)}{abc} = \frac{bc\sin(\pi - A)}{abc} = \frac{ca\sin(\pi - B)}{abc}$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

Hence $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (proved)

Ex-25

What interface can you draw when $\vec{a}\times\vec{b}$ and $\vec{a}\cdot\vec{b} = \vec{0}$.

Given $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}\cdot\vec{b} = \vec{0}$

(Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$) and

($\vec{a} = \vec{0}$ are $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$)

As $\vec{a} \parallel \vec{b}$ and $\vec{a} \perp \vec{b}$ cannot be hold simultaneously so $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

Hence either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

(26) If $|\vec{a}| = 2$ and $|\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, then find $\vec{a}\cdot\vec{b}$.

⇒ Given $|\vec{a} \times \vec{b}| = 8$

$\Rightarrow |\vec{a}||\vec{b}| \sin\theta = 8$

$2 \times 5 \sin\theta = 8$

$\Rightarrow \sin\theta = \frac{8}{10} = \frac{4}{5}$

$(\because) \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

(27) show that the vectors $\hat{i} - 3\hat{j} + 4\hat{k}$, $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} + 7\hat{j} + 10\hat{k}$ are co-planer.

Now let us find the following determinant

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) - (-3)(20-8) + 4(-14+4) =$$

$$= 4 + 36 - 40 = 0$$

Hence, the three given vectors are co-planer.

Limit :-

A limit tells us the value that a function approches as that function's inputs get closer and closer to same number.

i.e $\lim_{x \to a} f(x) = l$

* Functional value always gives the exact value of a function at a point where as limiting value gives an approximated value of function.
* Functional value is either defined or undefined similarley limiting value is either exist or doesn't exist.

Left hand limit :-

when x approches a from left then the value to which f(x) approches is called left hand limit of f(x) at x-a written as :

$L.H.L = \lim_{x \to a^-} f(x)$.

Right hand limit :-

where x approches a from right then the value to which f(x) approches is called right hand ~~value~~ limit. $R.H.L = \lim_{x \to a^+} f(x)$

$x \to a^+$ means $x \in (a, a+s)$.

Existency of limit :-

If L.H.L = R.H.L i.e

$$\boxed{\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l}$$

Then the limit of the function exists and $\lim_{x \to a} f(x) = l$.

otherwise limit doesn't exist.

ALGEBRA OF LIMIT :-

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$.

Then,

(i) $\lim_{x\to a}\{f(x)+g(x)\}=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=l+m$

(ii) $\lim_{x\to a}\{f(x)-g(x)\}=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=l-m$

(iii) $\lim_{x\to a}\{f(x).g(x)\}=\lim_{x\to a}f(x).\lim_{x\to a}g(x)=lm$

(iv) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{l}{m}$ (provided $m\neq0$)

(v) $\lim_{x\to a}k=k$ = k is constant.

(vi) $\lim_{x\to a}k\,f(x)=k\lim_{x\to a}f(x)=kl$

(vii) $\lim_{x\to a}\log_b f(x)=\log_b\lim_{x\to a}f(x)=\log_a l$

(viii) $\lim_{x\to a}e^{f(x)}=e^{\lim_{x\to a}f(x)}=e^l$

(ix) $\lim_{x\to a}f(x)^n=\{\lim_{x\to a}f(x)\}^n=l^n$

(x) $\lim_{x\to a}|f(x)|=\lim_{y\to a}f(y)=|l|$

(xi) $\lim_{x\to a}f(x)=\lim_{y\to a}f(y)=l$

(xii) If $\lim_{x\to a}f(x)=\infty$ then $\lim_{x\to a}\frac{1}{f(x)}=0$

EVALUTION OF LIMIT :-

When we evaluate limits it is not necessary to test the existency of limit always so in this section we will discuss various methods of evaluting limits.

(a) Evalution of Algebraic limits :-

(i) Direct substitution

(ii) Factorisation

(iii) Rationalisation.

(i) Direct substitution :-

If $f(x)$ is an algebraic function and $f(a)$ is finite. Then $\lim_{x\to a}f(x)$ is equal to $f(a)$ i.e we can substitute x by a.

Example -1

Evaluate: $\lim_{x\to 0}(x^2+2x+1)$

$\lim_{x \to 0} (x^2+2x+1) = 0^2+2(0)+1 = 1$

Ex-2

Evaluate $\lim_{x \to -1} \frac{x-1}{x^2+2x-1}$

$\lim_{x \to -1} \frac{x-1}{x^2+2x-1} = \frac{(-1)-1}{(-1)^2+2(-1)-1} = \frac{-2}{1-2-1} = \frac{-2}{-2} = 1$

Ex-3

$\lim_{x \to 1} \frac{\sqrt{x}}{\sqrt{x+2}}$

$\lim_{x \to 1} \frac{\sqrt{x}}{\sqrt{x+2}} = \frac{\sqrt{1}}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$

Ex-4 :-

Evalute $\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$ = which cannot be determined.

Note :-

so here direct substitution method fails to find the limiting value. in this case we apply following method.

(4) FACTORISATION METHOD :-

If the given function is a rational function $\frac{f(x)}{g(x)}$ and $\frac{f(a)}{g(a)}$ is in $\frac{0}{0}$ form.

then we apply Factorization method i.e we factorise $f(x)$ and $g(x)$ and cancel the common factor. After cancellitaion we again apply direct substitution. If result is a finite number. otherwise we repeat the process.

Ex- 4

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$

$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x^2+1)(x-1)}{x-1}$

$= \lim_{x\to 1} (x+1)$ $\{x\to 1$ means $x \neq 1 \Rightarrow (x-1) \neq 0\}$

$= 1+1 = 2$ (after cancletion we can apply the direct substitution)

Ex-5

Evaluate $\lim_{x\to -3} \frac{x^2+7x+12}{x^2+5x+6}$

Ans :-

$\lim_{x\to -3} \frac{x^2+7x+12}{x^2+5x+6}$ {by putting $x=-3$ we can easily check that the question is in $\frac{0}{0}$ form}

$= \lim_{x\to -3} \frac{x^2+4x+3x\cancel{+12}}{x^2+2x+3x+6}$

$= \lim_{x\to -3} \frac{x(x+4)+3(x+4)}{x(x+2)+3(x+2)}$

$= \lim_{x\to -3} \frac{(x+4)(x+3)}{(x+2)(x+3)}$ $\{x\to -3$ then $x+3 \neq 0\}$

$= \lim_{x\to -3} \frac{(x+4)}{(x+2)}$ $\frac{-3+4}{-3+2} = \frac{1}{-1} = 1$

Ex-6

Evaluate $\lim_{x \to 4} \frac{x^3 - 3x^2 - 3x - 4}{x^2 - 4x}$

As $x = 4$ gives $\frac{0}{0}$ form

$\Rightarrow x-4$ is factor of both polynomials

$$
\begin{array}{r|l|l}
x-4 & x^3 - 3x^2 - 3x - 4 & x^2 + x + 1 \\
 & x^3 - 4x^2 & \\
 & (-) \quad (+) & \\
\hline
 & x^2 - 3x - 4 & \\
 & x^2 - 4x & \\
 & (-) \quad (+) & \\
\hline
 & x - 4 & \\
 & x - 4 & \\
\hline
 & 0 &
\end{array}
$$

Hence $x^3 - 3x^2 - 3x - 4 = (x-4)(x^2+x+1)$

Now $\lim_{x \to 4} \frac{x^3 - 3x^2 - 3x - 4}{x^2 - 4x} = \frac{(x-4)(x^2+x+1)}{x^2 - 4x}$

$= \lim_{x \to 4} \frac{x^2 + x + 1}{x}$

$= \frac{4^2 + 4 + 1}{4}$

$= \frac{21}{4}$

(iii) <u>Rationalisation Method</u> :-

when either the number or the denominator contain some irrational function and direct substitution gives $\frac{0}{0}$ from, then we apply rationlization. in this method we rationalize the irrational function to eliminate the $\frac{0}{0}$ form.

Ex-7

Evaluate $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$

Ans:-

$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$ {in order to rationalize $\sqrt{x+1}-1$ we have to apply a^2-b^2 formula $a^2-b^2 = (a+b).(a-b)$ so here $a-b$ is present, so we have to multiply $a+b$ i.e $\sqrt{x+1}+1$ }

$\Rightarrow \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{\sqrt{x+1}^2-1^2}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x+1-1}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x}$

$= \lim_{x\to 0} \sqrt{x+1}+1 = \sqrt{0+1}+1 = 1+1 = 2$

Ex-8 :-

Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}\ \sqrt{1-x}}{2x}$

Ans: $\lim_{x\to 0} \frac{\sqrt{1+x}\ \sqrt{1-x}}{x} = \frac{0}{0}$ form

$= \lim_{x\to 0} \left(\frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{2x(\sqrt{1+x} + \sqrt{1-x})} \right)$

$= \lim_{x\to 0} \left(\frac{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}{2x(\sqrt{1+x} + \sqrt{1-x})} \right)$

$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{2x(\sqrt{1+x}) + (\sqrt{1-x})}$

$= \lim_{x\to 0} \frac{2x}{2x(\sqrt{1+x}) + (\sqrt{1-x})}$

$= \lim_{x\to 0} \frac{1}{1(\sqrt{1+x} + \sqrt{1-x}}$

$= \lim_{x\to 0} \frac{1}{\sqrt{1+0}\ \sqrt{1-0}}$

$= \frac{1}{1+1}$

$= \frac{1}{2}$ (Ans)

(3) Evaluting limit when $x \to \infty$

In order to evalute infite limits we use some formulas and techniques.

formula (i) $\lim_{x\to\infty} x^n = \infty, n > 0$

(ii) $\lim_{x\to\infty} \frac{1}{x^n} = 0, n > 0$

when we evalute functions in $\frac{f(x)}{g(x)}$ form, the we the following technique.

Divide both $f(x)$ and $g(x)$ by x^k where x^k is the highest order term in $g(x)$

Ex - 9

Evaluate $\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-7x+5}$

Ans :-

$$\lim_{x\to\infty} \frac{\frac{3x^2+x-1}{x^2}}{\frac{2x^2-7x+5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{7x}{x^2}+\frac{5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{1}{x}-\frac{1}{x^2}}{2-\frac{7}{x}+\frac{5}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{7}{x} + \lim_{x\to\infty}\frac{5}{x^2}}$$

$$= \frac{3+0+0}{2+0+0} = \frac{3}{2} \text{ (Ans)}$$

Ex - 10 :-

Evaluate $\lim_{x\to\infty} \frac{x^3+2x^2+3}{x^4-3x+1}$

Ans $\lim_{x \to \infty} \frac{x^3+2x^2+3}{x^4-3x+1}$

{Dividing numerator and denominator by highest order term x^4}

$$= \lim_{x \to \infty} \frac{\frac{x^3+2x^2+3}{x^4}}{\frac{x^4-3x+1}{x^4}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^4}}{1-\frac{3}{x^3}+\frac{1}{x^4}} = \frac{0+0+0}{1-0+0} = \frac{0}{1} = 0$$

Ex – 11

Evalute $\lim_{x \to \infty} \frac{x^2+5x+2}{x^3+2}$

Ans :-

$$\lim_{x \to \infty} \frac{x^4+5x+2}{x^3+2}$$

{Dividing numerator & denominator by highest order term of denominator i.e x^3}

$$= \lim_{x \to \infty} \frac{x^4+5x+2}{x^3+2}$$

$$= \lim_{x \to \infty} \frac{\frac{x^4+5x+2}{x^3}}{\frac{x^3+2}{x^3}} = \lim_{x \to \infty} \frac{x+\frac{5}{x^2}+\frac{2}{x^3}}{1+\frac{2}{x^3}}$$

$$= \lim_{x \to \infty} \frac{x-0+0}{1+0} = \lim_{x \to \infty} x = \infty$$

Ex-12

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$$

Ans: $\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{x}}{\frac{4x+3}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2-1}{x^2}} \quad \sqrt{\frac{2x^2-1}{x^2}}}{\frac{4x}{x}+\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2}{x^2} \quad \frac{1}{x^2}} \quad \sqrt{\frac{2x^2}{x^2} \quad \frac{1}{x^2}}}{4+\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3-\frac{1}{x^2}} \quad \sqrt{2-\frac{1}{x^2}}}{4+\frac{3}{x}}$$

$$= \frac{(3-0)^{\frac{1}{2}}-(2-10)^{1/2}}{4+0}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4} \text{ (Ans)}$$

1) $\lim\limits_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$

2) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

3) $\lim\limits_{x \to} \frac{e^{x-1}}{x} = 1$

4) $\lim\limits_{x \to 0} (1+x)^{1/x} = e$

5) $\lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$

6) $\lim\limits_{x \to 0} \frac{\log(1+x)}{x} = 1$

7) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

8) $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

Substitution method :-

In order to apply known formula sometimes we apply substitution method. In this method x is replaced by another variable u and then we apply formula on u.

Example - 14 :-

Evaluate $\lim_{x \to 0} \frac{\sin 2x}{x}$.

Ans let $2x = u \Rightarrow$ when $x \to 0$

$u \to 0$ (as $u = 2x$)

Now $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin u}{\frac{u}{2}} = 2 \lim_{u \to 0} \frac{\sin u}{u}$

$= 2 \times 1 = 2$

In general putting $\lambda x = u$

$$\boxed{\begin{aligned} \lim_{x \to 0} f(\lambda x) &= \lim_{u \to 0} f(u) \\ &= \lim_{x \to 0} f(x) \end{aligned}}$$

Hence some of the formula may be stated as flows.

(1) $\lim_{x \to 0} \frac{a^{\lambda x} - 1}{\lambda x} = \log_e a$

in particular $\lim_{x \to 0} \frac{e^{\lambda x} - 1}{\lambda x} = 1$

(2) $\lim_{x \to 0} (1 + \lambda x)^{\frac{1}{\lambda x}} = e$

(3) $\lim_{x \to \infty} \left(1 + \frac{1}{\lambda x}\right)^{\lambda x} = e$

(4) $\lim_{x \to 0} \frac{\log_a (1 + \lambda x)}{\lambda x} = \log_a e$

(5) $\lim_{x \to 0} \frac{\sin \lambda x}{\lambda x} = 1$

(6) $\lim_{x \to 0} \frac{\tan \lambda x}{\lambda x} = 1$

some examples based on the formula.

(1) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\tan 5x}$

Ans $\lim_{x \to 0} \frac{\sin 3x}{\tan 5x} = \lim_{x \to 0} \frac{\frac{3\sin 3x}{3x}}{\frac{5\tan 5x}{5x}}$

$= \frac{3}{5} \lim_{x \to 0} \frac{\left(\frac{\sin 3x}{3x}\right)}{\left(\frac{\tan 5x}{5x}\right)} = \frac{3}{5} \times \frac{1}{1} = \frac{3}{5}$

(2) Evaluate $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

Ans :- $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}$

$= \lim_{x \to 0} 2 \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} = \frac{1}{2} \lim_{x \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}}$

$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

(3) Evalute $\lim_{x \to 0} \frac{e^{3x} - e^{x}}{x}$

$\lim_{x \to 0} \frac{e^{3x} - e^{x}}{x} = \lim_{x \to 0} \frac{e^{3x} - 1 + 1 - e^{x}}{x}$

$= \lim_{x \to 0} \frac{(e^{3x} - 1)}{x} - \lim_{x \to 0} \left(\frac{e^{x} - 1}{x}\right)$

$= \lim_{x \to 0} 3\left(\frac{e^{3x} - 1}{3x}\right) - \lim_{x \to 0} \frac{e^{x} - 1}{x} = 3 - 1 = 2$

(4) Evalute $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Ans :- $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin}{\cos}\right.$

$= \lim_{x \to \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x}\right)$

$= \lim_{x \to \frac{\pi}{2}} \left(\frac{(1 - \sin x)(1 + \sin x)}{\cos x\,(1 + \sin x)}\right)$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{1-\sin^2 x}{\cos x (1+\sin x)} \right) = \lim_{x \to \frac{\pi}{2}} \left(\frac{\cos^2 x}{\cos x (1+\sin x)} \right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\cos x}{1+\sin x} \right) = \frac{\cos \frac{\pi}{2}}{1+\sin \frac{\pi}{2}} = \frac{0}{1+1} = \frac{0}{2} = 0$$

(5) Evaluate $\lim_{x \to 0} \left(\frac{\tan x - \sin x}{\sin^3 x} \right)$

$$\lim_{x \to 0} \left(\frac{\tan x - \sin x}{\sin^3 x} \right) = \lim_{x \to 0} \left(\frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x - \sin x \cos x}{\sin^3 x \cos x} \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x (1-\cos x)}{\sin^3 x \cos x} \right)$$

$$= \lim_{x \to 0} \left(\frac{1-\cos x}{\sin^2 x \cos x} \right)$$

$$= \lim_{x \to 0} \left(\frac{1-\cos x}{(1-\cos^2 x)\cos x} \right)$$

$$= \lim_{x \to 0} \left(\frac{(1-\cos x)}{(1-\cos x)(1+\cos x)\cos x} \right)$$

$$= \lim_{x \to 0} \left(\frac{1}{(1+\cos x)\cos x} \right)$$

$$= \frac{1}{(1+\cos 0)\cos 0} = \frac{1}{(1+1) \cdot 1} = \frac{1}{2}$$

$\boxed{\text{Ans: } \frac{1}{2}}$

(6) Evalute $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$.

Ans :-

$$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{u \to 0} \frac{u}{\sin u}$$

{put $\sin^{-1} x = u \Rightarrow x = \sin u$ when $x \to 0$ $u \to \sin^{-1}$
$x \to 0$ { as $\sin^{-1} 0 = 0$ }}

$$= \lim_{u \to 0} \frac{1}{\frac{\sin u}{u}} = \frac{1}{1} = 1$$

(7) Evalute $\lim_{x \to 0} \frac{\tan^{-1} x}{x}$.

$$\lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{u \to 0} \frac{u}{\tan u}$$

{put $\tan^{-1} x = u \Rightarrow x = \tan u$ when $x \to 0$
$u \to 0$}

$$= \lim_{u \to 0} \frac{1}{\frac{\tan u}{u}} = 1/1 = 1$$

(8) Evalute $\lim_{x \to 2} \frac{x^3 - 8}{x^5 - 32}$

Ans:- $\lim_{x \to 2} \frac{x^3 - 8}{x^5 - 32} = \lim_{x \to 2} \frac{x^3 - 2^3}{x^5 - 2^5}$

{as $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$}

$$= \lim_{x \to 2} \frac{\frac{x^3 - 2^3}{x - 2}}{\frac{x^5 - 2^5}{x - 2}} = \frac{3}{5} \frac{2^{3-1}}{2^{5-1}} = \frac{3 \times 2^2}{5 \times 2^4}$$

$$= \frac{3}{20}$$

(9) Evaluate $\lim_{x \to 0} \dfrac{(3+x)^3 - 27}{x}$

$\lim_{x \to 0} \dfrac{(3+x)^3 - 27}{x}$ {put $x+3 = u$ when $x \to 0$ then $u \to 3$}

$= \lim_{u \to 0} \dfrac{u^3 - 3^3}{u-3}$

$= 3 \cdot 3^{3-1} = 3 \times 3^2 = 3 \times 9 = 27$

(10) Evaluate $\lim_{x \to 0} \dfrac{\tan^{-1} 3x}{\sin 7x}$

Ans - $\lim_{x \to 0} \dfrac{\tan^{-1} 3x}{\sin 7x} = \lim_{x \to 0} \dfrac{\frac{\tan^{-1} 3x}{3x} \cdot 3}{\frac{\sin 7x}{7x} \cdot 7}$

$= \dfrac{3}{7} \lim_{x \to 0} \left(\dfrac{\frac{\tan^{-1} 3x}{3x}}{\frac{\sin 7x}{7x}} \right)$

$= \dfrac{3}{7} \times \dfrac{1}{1} = \dfrac{3}{7}$

(11) Evaluate $\lim_{x \to 1} \dfrac{\log_e 2x-1}{x-1}$

{ for applying log formula $x \to 0$. but here $x \to 1$. so we have to substitute a new variable u as $u = x-1$ }

$= \lim_{u \to 0} \dfrac{\log_e 2(u+1)-1}{u}$

$= \lim_{u \to 0} \dfrac{\log_e 2u+1}{u}$ when $x \to 1$ then $u = x-1 \to 0$ }

$= \lim_{u \to 0} \frac{\log_e (1+2u)}{2u} \cdot 2 = 1 \times 2 = 2$

(12) Evalute $\lim_{x \to 0} \frac{4^x - 5^x}{3^x - 4^x}$

Ans :- $\lim_{x \to 0} \frac{4^x - 5^x}{3^x - 4^x} = \lim_{x \to 0} \frac{\frac{4^x - 1 + 1 - 5^x}{x}}{\frac{3^x - 1 + 1 - 4^x}{x}}$

$= \lim_{x \to 0} \frac{\frac{(4^x - 1)(5^x - 1)}{x}}{\frac{(3^x - 1)(4^x - 1)}{x}} = \lim_{x \to 0} \frac{\left(\frac{4^x - 1}{x}\right)\left(\frac{5^x - 1}{x}\right)}{\left(\frac{3^x - 1}{x}\right)\left(\frac{4^x - 1}{x}\right)}$

$= \frac{\log_e 4 - \log_e 5}{\log_e 3 - \log_e 4} = \frac{\ln 4 - \ln 5}{\ln 3 - \ln 4} = \frac{\ln \frac{4}{5}}{\ln \frac{3}{4}}$

(13) Evaluate $\lim_{x \to 0} (1 + 3x)^{\frac{1}{x}}$

$\lim_{x \to 0} (1 + 3x)^{\frac{1}{x}} = \lim_{x \to 0} \{ 1 + 3x \}^{\frac{1}{3x} \cdot 3}$

$= \{ \lim_{x \to 0} (1 + 3x)^{\frac{1}{3x}} \}^3 = e^3$

(14) Evaluate $\lim_{x \to 0} \left(1 + \frac{2x}{3}\right)^{\frac{1}{2x}}$

Ans :

$\lim_{x \to 0} \left(1 + \frac{2x}{3}\right)^{\frac{1}{2x}} = \lim_{x \to 0} \left(1 + \frac{2x}{3}\right)^{\frac{2}{3x} \cdot 1/3}$

$= \{ \lim_{x \to 0} \left(1 + \frac{2x}{3}\right)^{\frac{1}{\left(\frac{2x}{3}\right)}} \}^{1/3} = e^{1/3}$

USE OF L.H.L and R.H.L to Find limit of a Function

L.H.L and R.H.L used to find limit of a function where the defination of a function changes. for example $|x|$ at 0 or $[x]$ at any integral point etc.

Also the same concept is used when we come across following terms.

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$$
$$\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$$

Examples -

1) Evaluate $\lim_{x\to 0} \frac{|x|}{x}$

Ans $L.H.L = \lim_{x\to 0^-} \frac{|x|}{x}$

$= \lim_{x\to 0^-} \frac{-x}{x}$

$= \lim_{x\to 0^-} (-1) = -1$

$R.H.L = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

From above

$L.H.L \neq R.H.L \Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ doesn't exist.

(2) Evaluate $\lim_{x\to -1} \frac{x+1}{|x+1|}$

$\lim_{x\to -1} \frac{x+1}{|x+1|} = \lim_{u\to 0} \frac{u}{|u|}$ {let $x+1 = u$ when $x\to -1$ then $u\to 0$}

$L.H.L = \lim_{u\to 0^-} \frac{u}{|u|} = \lim_{u\to 0} \frac{u}{-u}$ {$u\to 0^- \Rightarrow u<0 \Rightarrow |u| = -u$}

$= \lim_{u\to 0} (-1) = 1$

$R.H.L = \lim_{u\to 0^+} \frac{u}{|u|} = \lim_{u\to 0^+} \frac{u}{u} = \lim_{u\to 0^+} 1 = 1$

Hence $\lim_{u\to 0} \frac{u}{|u|}$ does exist

(3) Find $\lim_{x\to 0^+} \{[x]+10\}$

$= \lim_{x\to 0^+} \{[x]+10\}$

$= \lim_{x\to 0^+} (0+10)$ {As $x\to 0^+ \Rightarrow x\in(0,1)$ i.e $0<x<1$} $\Rightarrow [x]=0$}

$= \lim_{x\to 0^+} 10 = 10$.

(4) Find $\lim_{x\to 3.7} [x]$

$\lim_{x\to 3.7} [x] = [3.7] = 3$

Use of L.H.L and R.H.L to find limit of a function

L.H.L and R.H.L used to find limit of a function where the defination of a function changes for example $|x|$ at 0 or $[x]$ at any integral point etc.

Also the same concept is used when we come across following terms.

$$\boxed{\begin{array}{l} \lim_{x \to 0^+} \frac{1}{x} = \infty \\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \\ \lim_{x \to 0^+} e^{\frac{1}{x}} = \infty \\ \lim_{x \to 0^-} e^{\frac{1}{x}} = 0 \end{array}}$$

Examples -

1) Evaluate $\lim_{x \to 0} \frac{|x|}{x}$

Ans L.H.L $= \lim_{x \to 0^-} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{-x}{x}$

$= \lim_{x \to 0} (-1) = -1$

R.H.L $= \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$

From above

L.H.L $\neq$ R.H.L $\Rightarrow \lim_{x \to 0} \frac{|x|}{x}$ doesn't exist.

(2) Evaluate $\lim_{x \to -1} \frac{x+1}{|x+1|}$

$\lim_{x \to -1} \frac{x+1}{|x+1|} = \lim_{u \to 0} \frac{u}{|u|}$ {let $x+1 = u$ when $x \to -1$ then $u \to 0$}

L.H.L $= \lim_{u \to 0^-} \frac{u}{|u|} = \lim_{u \to 0} \frac{u}{-u}$ {$u \to 0^- \Rightarrow u < 0 \Rightarrow |u| = -u$}

$= \lim_{u \to 0} (-1) = 1$

R.H.L. $\lim_{u \to 0^+} \frac{u}{|u|} = \lim_{u \to 0^+} \frac{u}{u} = \lim_{u \to 0^+} 1 = 1$

Hence $\lim_{u \to 0} \frac{u}{|u|}$ does exist

(3) Find $\lim_{x \to 0^+} \{[x] + 10\}$

$= \lim_{x \to 0^+} \{[x] + 10\}$

$= \lim_{x \to 0^+} (0 + 10)$ {As $x \to 0^+ \Rightarrow x \in (0,1)$ i.e $0 < x < 1$} $\Rightarrow [x] = 0$}

$= \lim_{x \to 0^+} 10 = 10$.

(4) Find $\lim_{x \to 3.7} [x]$

$\lim_{x \to 3.7} [x] = [3.7] = 3$

5) Find $\lim_{x\to -1}[x]$

LHL $= \lim_{x\to -1^-}[x] = \lim_{x\to -1^-}(-2) = 2$

$\{$ As $x\to -1^- \Rightarrow x \in (-1-\delta, -1)$, i.e. $-2 < x < -1 \Rightarrow [x] = -2\}$

R.H.L. $= \lim_{x\to -1^+}[x] = \lim_{x\to -1^+} -1 = 1$

As from above L.H.L $\neq$ R.H.L

$\Rightarrow \lim_{x\to -1}[x]$ does not exist.

(6) Evaluate $\lim_{x\to \frac{4}{3}}[3x-1]$

$\lim_{x\to \frac{4}{3}}[3x-1] = \lim_{u\to 3}[u]$

$\{$put $3x-1=u \Rightarrow$ when $x\to \frac{4}{3}$, $u\to 3\times\frac{4}{3}-1$ i.e. $u\to 3\}$

Now L.H.L $= \lim_{u\to 3^-}[u] = \lim_{u\to 3^-} 2 = 2$ $\{$As $u\to 3^- \Rightarrow 2<u<3 \Rightarrow [u]=2\}$

R.H.L $= \lim_{u\to 3^+}[u] = \lim_{u\to 3^+} = 3$

Hence L.H.L $\neq$ R.H.L

Therefore $\lim_{x\to \frac{4}{3}}[3x-1]$ does not exist.

(7) Evaluate $\lim_{x\to 2} f(x)$ where

$$f(x) \begin{cases} -x & x<1 \\ x+1 & x\geq 1. \end{cases}$$

As $f(x)$ doesn't changes its defination at "2" so

$\lim_{x\to} f(x) = \lim_{x\to 2}(x+1) = 2+1 = 3$

$\{$As $x\to 2 \Rightarrow x\in(2-\delta, 2+\delta) \Rightarrow x>1$, $= \cancel{f(x)} =$ $\{ \cancel{x^2} \; \cancel{x<1}$, $\cancel{2x+1} \; \cancel{1\leq x\leq 2}$, $5 \; x\geq 2$

(8) Evaluate $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ if $f(x)$

$$= \begin{cases} x^2 & x<1 \\ 2x+1 & 1\leq x\leq 2 \\ 5 & x>2 \end{cases}$$

L.H.L $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$

R.H.L $= \lim_{x\to 1^+} f(x)$ $\lim_{x\to 1^+} 2x+1 = 2\times 1 + 1 = 3$

$\{$when $x\to 1^- \Rightarrow x<1$ so we use $f(x) = x^2\}$

$\{$ when $x\to 1^+ \Rightarrow x>$ i.e. $1<x<2 \Rightarrow f(x) = 2x+1\}$

From above LHL $\neq$ R.HL

$\Rightarrow \lim_{x\to 1} f(x)$ doesn't exist.

$\lim\limits_{x\to 2} f(x)$

L.H.L $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+1) = 2\times 2+1 = 5$

{when $x\to 2^- \Rightarrow x \le 1 \le x \le 2$ i.e. $1 < x < 2 \Rightarrow f(x) = 2x+1$ }

R.H.L $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5 = 5$

{ $x\to 2^+ \Rightarrow x>2 \Rightarrow f(x) = 5$ from defination }

As LHL = RHL.

Therefore $\lim_{x\to 2} f(x) = 5$

(9) Evaluate $\lim_{x\to 0} \frac{1}{x}$

L.HL $= \lim_{x\to 0^-} \frac{1}{x} = -\infty$

R.HL $= \lim_{x\to 0^+} \frac{1}{x} = \infty$

L.HL $\neq$ R.HL

$\Rightarrow \lim_{x\to 0} \frac{1}{x}$ doesnt exist.

__Note__

$$x \to a \begin{cases} x\to a^- \\ x\to a^+ \end{cases}$$

so when we use direct substitution method either for $x\to a^+$ or $x\to a^-$ in both case we have to replace x by a.

__Sandwhich theorm :-__

$$\boxed{\lim_{x\to a} h(x) = 1}$$

__Example :-__

find $\lim_{x\to 0} x \sin\frac{1}{x}$

__solution__ : we know $|\sin\frac{1}{x}| \le 1$

$\Rightarrow |x \sin\frac{1}{x}| \le |x|$

Again $|x \sin\frac{1}{x}| \ge 0$

Hence $0 \le |x \sin\frac{1}{x}| \le |x|$

Now $\lim_{x \to 0} 0 = 0$

And $\lim_{x \to 0} |x| = 0$

{As $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = -0 = 0$ and $\lim_{x \to 0^+} |x|$
$= \lim_{x \to 0^+} x = 0$}

Hence by sandwich theorm

$\lim_{x \to 0} |x \sin \frac{1}{x}| = 0$

when $x \to 0$, $x \sin \frac{1}{x} = (+)ve$ so $|x \sin \frac{1}{x}| = x \sin \frac{1}{x}$

{when $x \to 0^-$ then $x \in (\delta, 0) = (-v)e$ $\sin \frac{1}{x} = -ve \Rightarrow$
$x \sin \frac{1}{x} = +ve$},

{when $x \to 0^+$ the $x \in (0\delta)$ $x = +ve$ $\sin \frac{1}{x}$ $+ve \Rightarrow$
$x \sin \frac{1}{x} = +ve$}

Hence $\boxed{\lim_{x \to 0} x \sin \frac{1}{x} = 0}$

<u>illustrative Examples:-</u>

1) Evaluate $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$ $a, b \neq 0$

Ans:- $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \dfrac{\frac{\sin ax}{ax} \cdot a}{\frac{\sin bx}{bx} \cdot b}$

$= \frac{a}{b} \lim_{x \to 0} \dfrac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)}$

$= \frac{a}{b} \times \frac{1}{1} = \frac{a}{b}$

2. Evaluate $\lim_{x\to 0} \frac{x - x\cos 2x}{\sin^3 2x}$

$\lim_{x\to 0} \frac{x - x\cos 2x}{\sin^3 2x} = \lim_{x\to 0} \frac{x(1-\cos 2x)}{\sin^3 2x}$

$= \lim_{x\to 0} \frac{x\,2\sin^2 x}{\sin^3 2x} = 2\lim_{x\to 0} \frac{\frac{x\sin^2 x}{x^3}}{\frac{\sin^3 2x}{(2x)^3} 2^3}$

$= 2\lim_{x\to 0} \frac{\frac{\sin^2 x}{x^2}}{\frac{\sin^3 2x}{(2x)^3}\cdot 8}$

$= \frac{2}{8}\lim_{x\to 0} \frac{\left(\frac{\sin x}{x}\right)^2}{\left(\frac{\sin 2x}{2x}\right)^3}$

$= \frac{1}{4} \times \frac{1^2}{1^2} = \frac{1}{4}$ (Ans)

(3.) Evaluate $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2}$

$\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} = \lim_{x\to 0} \frac{2\sin\left(\frac{mx+nx}{2}\right)\sin\frac{nx-mx}{2}}{x^2}$

$= \lim_{x\to 0} \frac{2\sin\left(\frac{n+m}{2}\right)x}{x} \cdot \frac{\sin\left(\frac{n-m}{2}\right)x}{x}$

$= \lim_{x\to 0} 2\left(\frac{m+n}{2}\right) \frac{\sin\left(\frac{m+n}{2}\right)x}{\left(\frac{m+n}{2}\right)x} \left(\frac{n-m}{2}\right) \frac{\sin\left(\frac{n-m}{2}\right)x}{\left(\frac{n-m}{2}\right)x}$

$= 2\left(\frac{m+n}{2}\right)\left(\frac{n-m}{2}\right)$

$\lim_{x\to 0} \frac{\sin\left(\frac{m+n}{2}\right)x}{\left(\frac{m+n}{2}\right)x} \cdot \frac{\sin\frac{n-m}{2}x}{\left(\frac{n-m}{2}\right)x}$

$= 2\frac{(m+n)}{2}\frac{(n-m)}{2} \times 1 \times 1$

$= \frac{(m+n)(n-m)}{2}$

$= \frac{n^2-m^2}{2}$

4. Evaluate $\lim_{x\to\frac{\pi}{2}} (\frac{\pi}{2}-x)\tan x$

$\lim_{x\to\frac{\pi}{2}} (\frac{\pi}{2}-x)\tan x = \lim_{u\to 0} u\tan(\frac{\pi}{2}-u)$

$= \lim_{u\to 0} u\cot u$

$= \lim_{u\to 0} \frac{u}{\tan u}$

$= \lim_{u\to 0} \frac{u/u}{\tan u/u}$

$= \lim_{u\to 0} \frac{1}{(\frac{\tan u}{u})} = \frac{1}{1} = 1$

(5) Evaluate $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-x}$

$\lim_{x\to 1} \frac{x^2-2x+1}{x^2-x} = \lim_{x\to 1} \frac{(x-1)^2}{x(x-1)}$

$= \lim_{x\to 1} \frac{x-1}{x}$

$= \frac{1-1}{1} = \frac{0}{1} = 0$

(6) Evaluate $\lim_{x\to a} \frac{\sqrt{x-b}\ \sqrt{a-b}}{x^2-a^2}$, $(a>b)$

$\lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$

$= \lim_{x\to a} \frac{(\sqrt{x-b}-\sqrt{a-b})(\sqrt{x-b}+\sqrt{a-b})}{(x^2-a^2)(\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a} \frac{(x-b)-(a-b)}{(x-a)(x+a)(\sqrt{x-b}\sqrt{a-b})}$

$= \lim_{x\to a} \frac{x-b-a+b}{(x-a)(x+a)\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a} \frac{(x-a)}{(x-a)\cdot(x+a)\sqrt{x-b}+\sqrt{a+b})}$

$\lim_{x\to a} \frac{1}{(x+a)(\sqrt{x-b}+\sqrt{a-b})} = \frac{1}{(a+a)(\sqrt{a-b}+\sqrt{a-b})}$

$= \frac{1}{2a\,2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-b}}$

(7) Evaluate $\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$

$\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$

{put $2x-1=u \Rightarrow$ when $x \to \frac{1}{2}$, $u \to 2x\frac{1}{2}-1=0$}

$= \lim_{u\to 0} \frac{|u|}{u}$

L.H.L $= \lim_{u\to 0^-} \frac{|u|}{u} = \lim_{u\to 0} \frac{-u}{u} = \lim_{u\to 0^-}(-1) = -1$

R.H.L $= \lim_{u\to 0^+} \frac{|u|}{u} = \lim_{u\to 0^+} \frac{u}{u} = \lim_{u\to 0^+} 1 = 1$

As LHL $\neq$ R.HL. so $\lim_{u\to 0^+} \frac{|u|}{u}$ doesn't exist

$\Rightarrow \lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$ doesn't exist.

(8) Evaluate $\lim_{x\to\infty} \frac{x}{[x]}$

from defination of $[x]$ we know that

$x-1 < [x] \leq x$

$\Rightarrow \frac{x}{x-1} > \frac{x}{[x]} \geq \frac{x}{x}$

$\Rightarrow 1 \leq \frac{x}{[x]} < \frac{x}{x-1}$

Now. $\lim_{x\to\infty} 1 = 1$

$\lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x-1}{x}} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = \frac{1}{1-0} = 1$

Hence by sandwhich therom $\lim_{x\to\infty}$

$\frac{x}{[x]} = 1$

(9) Evaluate $\lim_{n\to\infty} \frac{1^2+2^2+3^3+ \cdots +n^3}{n^3}$

$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^3}{n^3}$

$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$

$= \frac{1}{6} \lim_{n\to\infty} \frac{n}{n}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$

$= \frac{1}{6} \lim_{n\to\infty} 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{1}{6} \times 1(1+0)(2+0) = \frac{2}{6} = \frac{1}{3}$

(10) Evaluate $\lim_{x\to a} \frac{x\sin\alpha - \alpha\sin x}{x-\alpha}$.

$\lim_{x\to\alpha} \frac{x\sin\alpha - \alpha\sin x}{x-a}$ put $x-\alpha = u$ when $x \to \alpha$ the $u \to 0$

$= \lim_{u\to 0} \frac{(\alpha+u)\sin\alpha - \alpha\sin(u+\alpha)}{u}$

$= \lim_{u\to 0} \frac{(\alpha\sin\alpha + u\sin\alpha - \alpha\sin u\cos\alpha - \alpha\cos u\sin\alpha)}{u}$

$= \lim_{u\to 0} \frac{(\alpha\sin\alpha - \alpha\cos u\sin\alpha + u\sin\alpha - \alpha\sin u\cos\alpha)}{u}$

$= \lim_{u\to 0} \left\{ \frac{\alpha\sin\alpha(1-\cos u)}{u} + \sin\alpha - \alpha\cos\alpha \frac{\sin u}{u} \right\}$

$= \lim_{u\to 0} \left\{ \frac{\alpha\sin\alpha\, 2\sin^2\frac{u}{2}}{u} + \sin\alpha - \alpha\cos\alpha \frac{\sin u}{u} \right\}$

$= \lim_{u\to 0} \quad 2\alpha\sin\alpha \frac{\sin\frac{u}{2}}{\frac{u}{2}\cdot 2} \cdot \sin\frac{u}{2} + \sin\alpha - \alpha\cos\alpha \frac{\sin u}{u} \}$

$= \lim_{u\to 0} \left\{ \alpha\sin\alpha \frac{\sin\frac{u}{2}}{\frac{u}{2}} \cdot \sin\frac{u}{2} + \sin\alpha - \alpha\cos\alpha \frac{\sin u}{u} \right\}$

$= \alpha\sin\alpha \cdot 1 \cdot 0 + \sin\alpha - \alpha\cos\alpha \cdot 1 = \sin\alpha - \alpha\cos\alpha$

11) Evaluate $\lim_{x\to 5} \frac{\log_e^x - \log_e^5}{x-5}$

$\lim_{x\to 5} \frac{\log_e^x - \log_e^5}{x-5}$

$= \lim_{u\to 0} \frac{\log_e(u+5) - \log_e 5}{u}$

$= \lim_{u\to 0} \frac{\log_e \left(\frac{u+5}{5}\right)}{u}$

$= \lim_{u\to 0} \frac{\log_e\left(\frac{u}{5}+1\right)}{u}$

$= \lim_{u\to 0} \frac{\log_e\left(\frac{u}{5}+1\right)}{\frac{u}{5}\cdot 5}$

$= \frac{1}{5} \lim_{u\to 0} \frac{\log_e\left(\frac{u}{5}+1\right)}{\frac{u}{5}\cdot 5}$

$= \frac{1}{5}(\cdot 1) = \frac{1}{5}$

(12) Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\log_e(1+x)}$

$\lim_{x\to 0} \frac{\sqrt{1+x-1}}{\log_e(1+x)}$

$= \lim_{x\to 0} \frac{(\sqrt{1+x}-1)\cdot(\sqrt{1+x}+1)}{\log_e(1+x)\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1+x-1}{\log_e(1+x)(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1}{\frac{\log_e(1+x)}{x}(\sqrt{1+\cancel{x}}+1}$

$= \frac{1}{1\times\sqrt{1+0}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

(13) Evaluate $\lim_{x\to 2} \frac{\log_7(2x-3)}{x-2}$

$\lim_{x\to 2} \frac{\log_7(2x-3)}{x-2}$

$= \lim_{u \to 0} \frac{\log_7 \{2(u+2) - 3\}}{u} = \lim_{u \to 0} \frac{\log_7(2u+1)}{u}$

$= \lim_{u \to 0} \frac{2 \log_7 (1+2u)}{2u}$

$= 2 \lim_{u \to 0} \frac{\log_7 (1+2u)}{2u} = 2 \cdot \log_7 e$

(14) find the value of a for which

$$\lim_{x \to 1} \frac{5^x - 5}{(x-1) \log_e a} = 5$$

Given $\lim_{x \to 1} \frac{5^x - 5}{(x-1) \log_e a} = 5$ ——— (1)

Now $\lim_{x \to 1} \frac{5^x - 5}{(x-1) \log_e a}$

$= \lim_{u \to 0} \frac{5^{u+1} - 5}{u \log_e a}$

$= \lim_{u \to 0} \frac{5^u \cdot 5 - 5}{u \log_e a}$

$= \frac{5}{\log_e a} \lim_{u \to 0} \frac{5^u - 1}{u}$

$= \frac{5}{\log_e a} \log_e 5$ ——— (2)

= from (1) and (2) we have

$\frac{5}{\log_e a} \log_e 5 = 5$

$\Rightarrow \log_e 5 = \log_e a$

$\Rightarrow a = 5$

(15) Evaluate $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 6x + 5}$

$\lim_{x \to 1} \frac{x^2 - 3x - x + 3}{x^2 - 5x - x + 5} = \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 1} \frac{x(x-3)-1(x-3)}{x(x-5)-1(x-5)}$

$= \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-5)(x-1)}$

$= \lim_{x\to 1} \frac{x-3}{x-5}$

$= \frac{1-3}{1-5} = \frac{-2}{-4} = \frac{1}{2}$

(16) Evaluate $\lim_{x\to 0} \frac{3^x+3^{-x}-2}{x^2}$

$\lim_{x\to 0} \frac{3^x+3^{-x}-2}{x^2} = \lim_{x\to 0} \frac{3^x+\frac{1}{3^x}-2}{x^2}$

$= \lim_{x\to 0} \frac{3^{2x}+1-2\cdot 3^x}{3^x x^2} = \lim_{x\to 0} \frac{(3^x)^2-2\cdot 3^x\cdot 1+1^2}{3^x x^2}$

$= \lim_{x\to 0} \frac{(3^x-1)^2}{3^x x^2} = \lim_{x\to 0} \frac{1}{3^x}\left(\frac{3^x-1}{x}\right)^2$

$= \frac{1}{3^0}(\log_e 3)^2$

$= \ln 3^2$

(17) Evaluate $\lim_{x\to 0} \frac{e^{\tan x}-1}{x}$

$\lim_{x\to 0} \frac{e^{\tan x}-1}{x}$

$= \lim_{u\to 0} \frac{e^y-1}{\tan^{-1} y}$

$= \lim \frac{\frac{e^y-1}{y}}{\frac{\tan^{-1} y}{y}} = \frac{1}{1} = 1$

(19) Evaluate $\lim_{x \to \infty} \dfrac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 + 5x + 7}$.

$$\lim_{x \to \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 + 5x + 7} = \lim_{x \to \infty} \frac{\frac{3x^3}{x^3} - \frac{4x^2}{x^3} + \frac{6x}{3x} - \frac{1}{x^3}}{\frac{2x^3}{x^3} + \frac{x^2}{x^3} + \frac{5x}{x^3} + \frac{7}{x^3}}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{4}{x} + \frac{6}{x^2} - \frac{1}{x^3}}{2 + \frac{1}{x^2} + \frac{5}{x^2} + \frac{7}{x^3}} = \frac{3 - 0 + 0 - 0}{2 + 0 + 0 + 0} = \frac{3}{2}$$

(20) Evaluate $\lim_{x \to 2} \dfrac{\frac{1}{x^2} - \frac{1}{4}}{x - 2}$

$$\lim_{x \to 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x - 2} = \lim_{x \to 2} \frac{\frac{4 - x^2}{4x^2}}{x - 2}$$

$$= -\frac{1}{4} \lim_{x \to 2} \frac{x^2 - 4}{x^2(x-2)}$$

$$= \frac{1}{4} \lim_{x \to 2} \frac{(x-2)(x+2)}{x^2(x-2)}$$

$$= -\frac{1}{4}\left(\frac{2+2}{2^2}\right) = -\frac{1}{4} \text{ (Ans)}$$

Continuity of a function at a point.

Defination - A function $f(x)$ is said to be continuous at $x = a$, if it satisfies the following conditions.

(i) $\lim_{x \to a} f(x)$ exists

(ii) $f(a)$ is defined i.e finite.

(iii) $\lim_{x \to a} f(x) = f(a)$

Continuous functions:-

A function is said to be continuous if it is continuous at each point of its domain.

Examples:-

1) Examine the continuity of the function $f(x)$ at $x=3$

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & x \neq 3 \\ 6x & = 3 \end{cases}$$

$\lim_{x\to 3} f(x)$ $\lim_{x\to 3} \dfrac{x^2-9}{x-3} = \lim_{x\to 3} \dfrac{(x+3)(x-3)}{(x-3)}$

$= \lim_{x\to 3} (x+3) = 3+3 = 6$

From given data $f(3)=6$

Now from above $\lim_{x\to 3} f(x) = f(3)$

Therefore $f(x)$ is continous at $x=3$

(2) Test continuity of $f(x)$ at '0' where

$$f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & x \neq 0 \\ e^3 & x = 0 \end{cases}$$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+3x)^{\frac{1}{x}}$

$= \lim_{x\to 0} (1+3x)^{\frac{1}{3x}\cdot 3}$

$= \lim_{x\to 0} \{(1+3x)^{\frac{1}{3x}}\}^3$

$= \{\lim_{x\to 0} (1+3x)^{\frac{1}{3x}}\}^3$

$= e^3$

$\{$ as $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\lim_{x \to 0} (1+\lambda x)^{\frac{1}{\lambda x}} = e$

in particular $\lim_{x \to 0} (1+3x)^{\frac{1}{3x}} = e$

and we know $\lim_{x \to a} \{f(x)\}^n = \{\lim_{x \to a} f(x)\}^n$

from given data $f(0) = e^3$

Hence $\lim_{x \to 0} f(x) = f(0)$

Therefore, $f(x)$ is continous at $x = 0$

(3) Test continuity of $f(x)$ at $x = 0$

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\lim_{x \to a} f(x) = \lim_{x \to 0} \frac{|x|}{x}$

As $|x|$ is present and $x \to 0$ so we have to evaluate the above limit by L.H.L and R.H.L method.

L.H.L. $= \lim_{x \to 0} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{-x}{x}$

$= \lim_{x \to 0^-} (-1) = (-1)$

R.H.L $= \lim_{x \to 0} \frac{|x|}{x}$

$= \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = 1$

Hence L.H.L $\neq$ R.H.L.

Hence $f(x)$ is not continous at $x = 2$

(4) Test continuity of $f(x)$ at '0'

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 3x} & x \neq 0 \\ \frac{5}{3} & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 3x}{\tan 5x} = \lim_{x \to 0} \frac{\frac{\sin 3x}{x}}{\frac{\sin 5x}{x}}$

$= \lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3}{\frac{\tan 5x}{5x} \cdot 5}$

$= \frac{3}{5} \lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right) / \left(\frac{\tan 5x}{5x}\right)$

$= \frac{3}{5}\left(\frac{1}{1}\right) = \frac{3}{5}$

given that $f(0) = \frac{5}{3}$

Thus $\lim_{x\to 0} f(x) \neq f(0)$

Hence $f(x)$ is not continous at $x = 0$

(5) Test continuity of $\frac{x^2-4}{x-2}$ at $x = 2$

Here $f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$ undefined.

(6) Test continuity of $f(x)$ at $x = \frac{1}{2}$

$f(x) = \begin{cases} 1-x & x \leq 1/2 \\ x & x > 1/2 \end{cases}$

Ans: First understand the function properly

when $x < \frac{1}{2}$, $f(x) = 1-x$

$x > \frac{1}{2}$, $f(x) = x$

when $x = \frac{1}{2}$, $f(x) = 1 - x = 1 - \frac{1}{2} = \frac{1}{2}$

Now let us find the $\lim_{x\to 1/2} f(x)$

L.H.L $= \lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-} (1-x)$

$= 1 - \frac{1}{2} = \frac{1}{2}$

R.H.L $= \lim_{x\to \frac{1}{2}^+} f(x)$

$= \lim_{x\to\frac{1}{2}^+} x = \frac{1}{2}$

NOW from above L.H.L = R.H.L

$\Rightarrow \lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$ ———— (1)

from defination $f(\frac{1}{2}) = \frac{1}{2}$ ———— (2)

from (1) and (2)

$\lim_{x \to \frac{1}{2}} f(x) = f(\frac{1}{2})$

Hence, $f(x)$ is continous at $x = \frac{1}{2}$

Q7:- Test continuity of $F(x)$ at $x = 0$

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

Here given that

$f(x) = 2x+1$ for $x<0$ ———— (1)

when $x=0$ $f(x) = f(0) = 2x+1 = 2 \times 0 + 1 = 1$ ———— (2)

when $0<x<1$, $f(x) = x$ ———— (3)

when $x=1$, $f(x) = f(1) = x = 1$ ———— (4)

when $x>1$, $f(x) = 2x-1$ ———— (5)

continuity test at $x=0$

L.H.L $= \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} (2x+1)$

$= (2 \times 0) + 1 = 1$

R.H.L $= \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} x = 0$

~~the~~ As L.H.L $\neq$ R.H.L

$\Rightarrow \lim_{x \to 0} f(x)$ doesn't exist

Hence, $f(x)$ is not continous at $x=0$

continuity test at $x=1$

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x$

$= \lim_{x\to 1^-} x = 1$

R.H.L = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x-1$

$= 2\times 1 - 1 = 1$

L.H.L = R.H.L

$\lim_{x\to 1} f(x) = 1$

from given data $f(1) = 1$

Hence, $\lim_{x\to 1} f(x) = f(1)$

Therefore, $f(x)$ is continous at $x=1$

(8) Examine contunuity of $f(x) = [3x+11]$ at $x = \frac{11}{3}$

$\lim_{x\to -\frac{11}{3}} f(x) = \lim_{x\to -\frac{11}{3}} [3x+11]$

$= \lim_{u\to 0} [u]$ ——— (1)

Now $\lim_{u\to 0^-} [u] = \lim_{u\to 0^-} -1 = 1$

and $\lim_{u\to 0^+} [u] = \lim_{u\to 0^+} 0 = 0$

As L.H.L $\neq$ R.H.L

$\Rightarrow \lim_{u\to 0} [u]$ doesn't exist $\Rightarrow \lim_{x\to \frac{11}{3}}$

Hence, $f(x)$ is not contineous.

(9) Determine the value of k for which $f(x)$ is continous at $x=1$

$$f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x-1} & x \neq 1 \\ k & x = 1 \end{cases}$$

Given function is continous at $x=1$

$\Rightarrow \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \lim_{x\to 1} f(x) = k$ ——— (1)

Now, let us find $\lim_{x\to 1} f(x)$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$

$= \lim_{x \to 1} \frac{x^2 - 2x - x + 2}{x - 1}$

$= \lim_{x \to 1} \frac{x(x-2) - 1(x-2)}{x-1}$

$= \lim_{x \to 1} \frac{(x-2)(x-1)}{(x-1)}$

$= \lim_{x \to 1} (x-2) = 1 - 2 = -1$ ——— (2)

Hence, from (1) and (2) we have $k = -1$

Q) 10) if $f(x) = \begin{cases} ax^2 + b & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 2ax - b & \text{if } x > 1 \end{cases}$

is continous at $x = 1$ then find a and b.

Given that $f(x)$ is continous at $x = 1$

$\Rightarrow \lim_{x \to 1} f(x) = f(1)$

$\Rightarrow \lim_{x \to 1} f(x) = 1$

from (1) as $\lim_{x \to 1} f(x)$ ~~lim~~ exists

$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x)$

from (1) and (2) we have

$\lim_{x \to 1^-} f(x) = 1$

$\Rightarrow \lim_{x \to 1^-} (ax^2 + b) = 1$

$\Rightarrow a \times 1^2 + b = 1$

$\Rightarrow a + b = 1$ ——— 3

Again from (1) and (2)

$\lim_{x \to 1^+} (f(x)) = 1$

$\Rightarrow \lim_{x \to 1^+} (2ax - b) = 1$

$\Rightarrow (2 \times a \times 1) - b = 1$

$\Rightarrow 2a - b = 1$ ——— (4)

Eqn(3) $a+b=1$

Eqn(4) $2a-b=1$

$3a=2$

$\Rightarrow a=\frac{2}{3}$

from (3) $a+b=1$

$\Rightarrow b=1-a=1-\frac{2}{3}=\frac{1}{3}$

Hence $a\ \frac{2}{3}$ and $b=\frac{1}{3}$

(11) Find the value of 'a' such that

$$f(x)=\begin{cases}\frac{\sin ax}{\sin x} & x\neq 0\\ \frac{1}{a} & x=0\end{cases}$$

is continous at $x=0$

$f(x)$ is continous at $x=0$

$\Rightarrow \lim_{x\to 0} f(x)=f(0)$

$= \lim_{x\to 0} \frac{\sin ax}{\sin x}=\frac{1}{a}$

$= \lim_{x\to a} \frac{a\left(\frac{\sin ax}{\sin x}\right)}{\left(\frac{\sin x}{x}\right)}=\frac{1}{a}$

$\Rightarrow a\frac{1}{1}=\frac{1}{a}$

$\Rightarrow a^2=1$

$\Rightarrow a=+1$ (Ans)

(12) Examine the continuity of the fn

$$f(x)=\begin{cases}x^2\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$ at $x=0$

$Eq^n(3)$ $a+b=1$

$Eq^n(4)$ $2a-b=1$

$3a=2$

$\Rightarrow a=\frac{2}{3}$

from (3) $a+b=1$

$\Rightarrow b=1-a=1-\frac{2}{3}=\frac{1}{3}$

Hence $a\ \frac{2}{3}$ and $b=\frac{1}{3}$

(11) Find the value of 'a' such that

$$f(x)=\begin{cases}\frac{\sin ax}{\sin x} & x\neq 0\\ \frac{1}{a} & x=0\end{cases}$$

is continuous at $x=0$

$f(x)$ is continuous at $x=0$

$\Rightarrow \lim x\to 0\ f(x)=f(0)$

$\Rightarrow \lim x\to 0\ \frac{\sin ax}{\sin x}=\frac{1}{a}$

$$=\lim x\to a\ \frac{a\left(\frac{\sin ax}{\sin x}\right)}{\left(\frac{\sin x}{x}\right)}=\frac{1}{a}$$

$\Rightarrow a\frac{1}{1}=\frac{1}{a}$

$\Rightarrow a^2=1$

$\Rightarrow a=+1$ (Ans)

(12) Examine the continuity of the fn

$$f(x)=\begin{cases}x^2\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}\quad \text{at } x=0$$

let us evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

we know that $-1 \le \sin \frac{1}{x} \le 1$

$\Rightarrow (-1)\, x^2 \le x^2 \sin \frac{1}{x} \le x^2 \cdot 1$

$\Rightarrow -x^2 \le x^2 \sin \frac{1}{x} \le x^2$

Now $\lim_{x \to 0} (-x^2) = 0^2 = 0$

$\lim_{x \to 0} x^2 = 0^2 = 0$

Hence by sandwhich theorem

$\lim_{x \to 0} x^2 \sin \frac{1}{a} = 0$

given $f(0) = 0$

Hence $\lim_{x \to 0} f(x) = f(0)$

Therefore $f(x)$ is continous at $x = 0$

(3) Test continuity of $f(x)$ at $x = 0$

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Evaluation of $\lim_{x \to 0} f(x)$ is not possible directly.

$L.H.L = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$

$= \dfrac{0 - 1}{0 + 1} = -1$

$R.H.L = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$

$= \lim_{x \to 0} \dfrac{\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} - \dfrac{1}{e^{\frac{1}{x}}}}{\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} + \dfrac{1}{e^{\frac{1}{x}}}}$

$= \lim_{x \to 0^+} \dfrac{1 - \frac{1}{\frac{1}{e^x}}}{1 + \frac{1}{\frac{1}{e^x}}}$

$= \dfrac{1-0}{1+0} = 1$

From above L.H.L $\neq$ R.H.L

(Q19) Discuss the continuity of the fun.

$f(x) = \begin{cases} x - \dfrac{|x|}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$ at $x = 0$

L.H.L $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - \dfrac{|x|}{x}$

$= \lim_{x \to 0^-} \{x - \frac{|x|}{x}\}$

$= \lim_{x \to 0^-} \{x - (-1)\} = \lim_{x \to 0^-} \{x + 1\}$

$= 0 + 1 = 1$

R.H.L $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - \dfrac{|x|}{x}$

$= \lim_{x \to 0^+} \{x - \frac{x}{x}\}$

$= \lim_{x \to 0} \{x - 1\} = 0 - 1 = -1$

L.H.L $\neq$ R.H.L

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

Therefore, $f(x)$ is not continuous at $x = 0$.